Home work 5 Solutions

2 a) $$g_J = 1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)}$$

$S_{1/2}$: $j = 1/2$, $l = 0$, $g_J = 2$.

$P_{1/2}$: $j = 1/2$, $l = 1$, $g_J = \frac{2}{3}$.

$P_{3/2}$: $j = 3/2$, $l = 1$, $g_J = 4/3$.

b) choose product states as basis vectors

$J_z = 3/2$ $\qquad$ $1/2$ $\qquad$ $-1/2$ $\qquad$ $-\frac{3}{2}$

$|+\rangle \otimes |1,1\rangle \qquad |-\rangle \otimes |1,1\rangle, \ |+\rangle \otimes |1,0\rangle, \quad |-\rangle \otimes |1,0\rangle, \ |+\rangle \otimes |1,-1\rangle, \quad |-\rangle \otimes |1,-1\rangle$

$$H = H_{so} + H_{zeeman} = A\,\vec{S}\cdot\vec{L} + \gamma\,(L_z + 2S_z), \qquad \gamma = \frac{e}{2m} B_{ext}$$

$$H_{so} = \begin{bmatrix} \frac{A}{2} & & & & & \\ & -\frac{A}{2} & \frac{\sqrt{2}A}{2} & & & \\ & \frac{\sqrt{2}A}{2} & 0 & & & \\ & & & 0 & \frac{1}{\sqrt{2}}A & \\ & & & \frac{1}{\sqrt{2}}A & -\frac{A}{2} & \\ & & & & & \frac{A}{2} \end{bmatrix}, \quad H_{zeeman} = \begin{bmatrix} 2\gamma & & & & & \\ & 0 & 0 & & & \\ & 0 & \gamma & & & \\ & & & -\gamma & 0 & \\ & & & 0 & 0 & \\ & & & & & -2\gamma \end{bmatrix}$$

$$E_1 = \frac{A}{2} + 2\gamma, \qquad E_6 = \frac{A}{2} - 2\gamma.$$

$$E_{2,3} = \frac{\left(\gamma - \frac{A}{2}\right) \pm \sqrt{\gamma^2 + \gamma A + \frac{9}{4}A^2}}{2}$$

$$E_{4,5} = \frac{-\gamma - \frac{A}{2} \pm \sqrt{\gamma^2 - \gamma A + \frac{9}{4}A^2}}{2}.$$

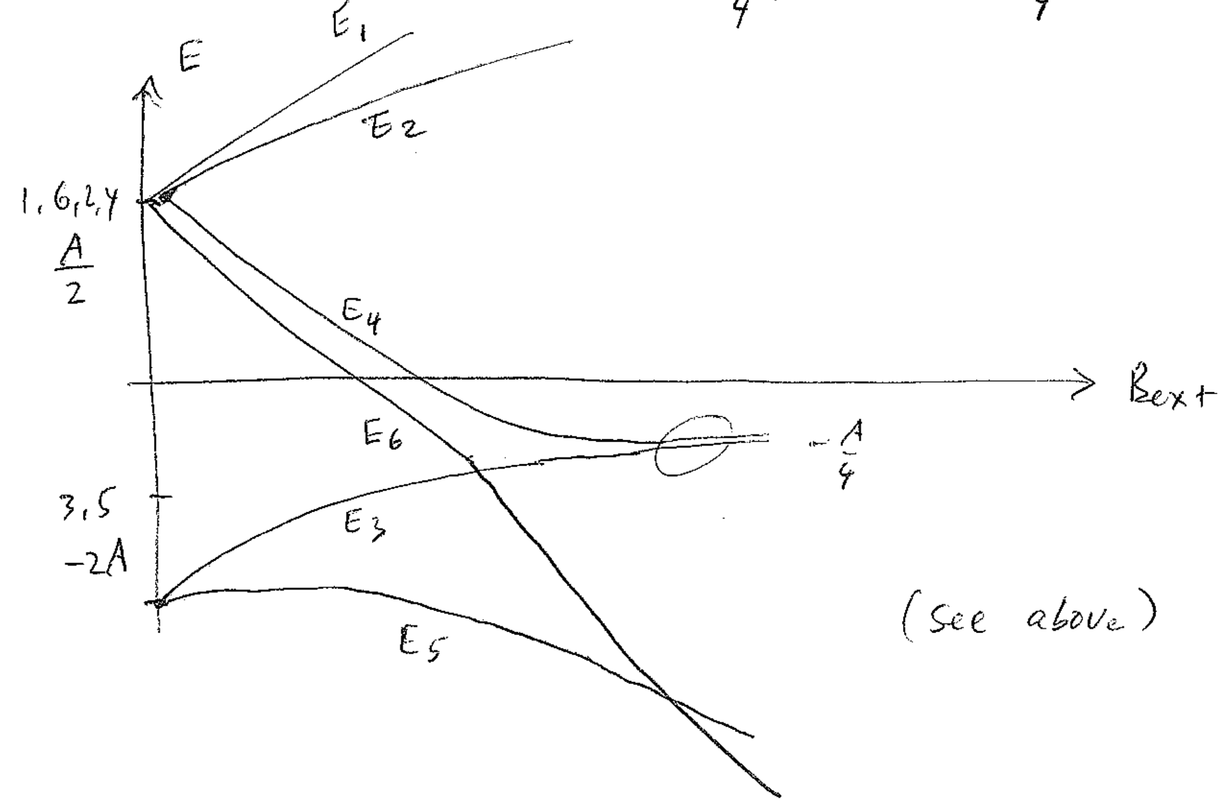
c) When $B_{ext} \to 0$, $E_1 = E_6 = E_2 = E_4 = \frac{A}{2}$

$E_3 = E_5 = -2A$ desired Result.

When $B_{ext} \to \infty$, $E_1 = 2\gamma$, $E_6 = -2\gamma$

$E_2 = \gamma - \frac{A}{4}$, $E_3 = -\frac{A}{2 \times 2} = -\frac{A}{4}$

$E_4 = -\frac{A}{4}$, $E_5 = -\gamma - \frac{A}{4}$



(see above)

Problem 6.38 (More)

deuteron $S=1$ states $|1,\pm1\rangle,\ |1,0\rangle$

electron $S=1/2$ states $|+\rangle,\ |-\rangle$

$S=3/2$ states are

$$\begin{cases} |+\rangle \otimes |1,1\rangle & S_z = \frac{3}{2} \\ \frac{1}{\sqrt{3}}\left(\sqrt{2}\,|+\rangle\otimes|1,0\rangle + |-\rangle\otimes|1,1\rangle\right), & S_z = \frac{1}{2} \\ \frac{1}{\sqrt{3}}\left(|+\rangle\otimes|1,-1\rangle + \sqrt{2}\,|-\rangle\otimes|1,0\rangle\right), & S_z = -\frac{1}{2} \\ |-\rangle\otimes|1,-1\rangle & S_z = -\frac{3}{2} \end{cases}$$

You can construct these states using lowering operators!

$S=1/2$ states are

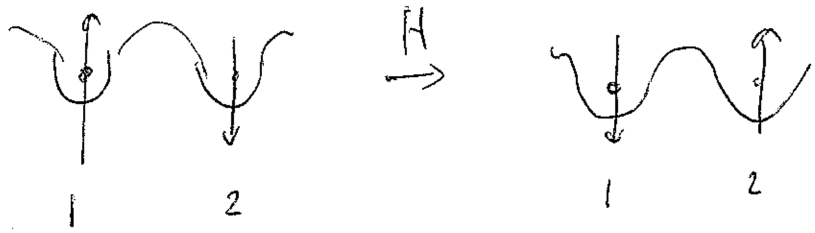
$$\begin{cases} \frac{1}{\sqrt{3}}\left(-|+\rangle\otimes|1,0\rangle + \sqrt{2}\,|-\rangle\otimes|1,1\rangle\right) & S_z = \frac{1}{2} \\ \frac{1}{\sqrt{3}}\left(-\sqrt{2}\,|+\rangle\otimes|1,-1\rangle + |-\rangle\otimes|1,0\rangle\right) & S_z = -\frac{1}{2} \end{cases}$$

almost identical to what we've done during lectures on Spin-orbit Coupling.

4)

a) $H = J\, \vec{S}_1 \cdot \vec{S}_2$

$$= J\left(S_{1z} S_{2z} + \frac{1}{2}\left(S_{1+} S_{2-} + S_{1-} S_{2+} \right) \right)$$



product states as basis vectors (Total 4 states

$$\underset{1}{|+\rangle} \otimes \underset{2}{|+\rangle} \qquad |+\rangle \otimes |-\rangle \qquad |-\rangle \otimes |+\rangle \qquad |-\rangle \otimes |-\rangle$$

$$H = \begin{bmatrix} \frac{J}{4} & & & \\ & -\frac{J}{4} & \frac{J}{2} & \\ & \frac{J}{2} & -J/4 & \\ & & & \frac{J}{4} \end{bmatrix}$$

$E_1 = \frac{J}{4}$ (triplet)

$$|+\rangle \otimes |+\rangle = |S=1, S_z=1\rangle$$

$$\frac{1}{\sqrt{2}}\left(|+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle \right) = |1, 0\rangle$$

$$|-\rangle \otimes |-\rangle = |1, -1\rangle$$

$E_2 = -\frac{3J}{4}$ (singlet) $\quad \frac{1}{\sqrt{2}}\left(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle \right) = |0, 0\rangle$

Alternatively, $H = J\left(\frac{S^2 - S_1^2 - S_2^2}{2}\right)$

$S = S_1 + S_2$, Total Spin. $[H, S^2] = 0$.

$S = 1, 0,$ $\begin{cases} S=1, & S_z = \pm 1, 0, \quad \text{triplet} \\ S=0, & S_z = 0 \quad \text{singlet} \end{cases}$

(explicit wavefunctions are given on page 4)

b)

$$|+\rangle \otimes |-\rangle = \frac{1}{\sqrt{2}} \left(\underbrace{|1,0\rangle}_{\text{triplet}} + \underbrace{|0,0\rangle}_{\text{singlet}} \right)$$

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}} \left(|1,0\rangle e^{-i\frac{J}{4\hbar}t} + |0,0\rangle e^{+i\frac{3J}{4\hbar}t} \right)$$

$$\boxed{P_{|-\rangle\otimes|+\rangle} = \sin^2\left(\frac{Jt}{2\hbar}\right)} \Leftarrow \left| \langle \Psi(t) | |-\rangle \otimes |+\rangle \rangle \right|^2$$

typical full oscillations between $|+\rangle \otimes |-\rangle$ and $|-\rangle \otimes |+\rangle$ states due ~~to~~ exchange interactions.

Prob. 1)

Problem 6.20

Equation 6.59 $\Rightarrow \mathbf{B} = \frac{1}{4\pi\epsilon_0}\frac{e}{mc^2r^3}\mathbf{L}$. Say $L = \hbar$, $r = a$; then

$$B = \frac{1}{4\pi\epsilon_0}\frac{e\hbar}{mc^2a^3}$$
$$= \frac{(1.60\times10^{-19}\,\mathrm{C})(1.05\times10^{-34}\,\mathrm{J\cdot s})}{4\pi\left(8.9\times10^{-12}\,\mathrm{C^2/N\cdot m^2}\right)(9.1\times10^{-31}\,\mathrm{kg})\left(3\times10^8\,\mathrm{m/s}\right)^2\left(0.53\times10^{-10}\,\mathrm{m}\right)^3} = \boxed{12\text{ T.}}$$

So a "strong" Zeeman field is $B_{\text{ext}} \gg 10$ T, and a "weak" one is $B_{\text{ext}} \ll 10$ T. Incidentally, the earth's field (10^{-4} T) is definitely *weak*.

Prob. 3

Problem 6.38

$$\text{Equation 6.89} \Rightarrow E^1_{\text{hf}} = \frac{\mu_0 g_d e^2}{3\pi m_d m_e a^3}\langle \mathbf{S}_d\cdot\mathbf{S}_e\rangle;\quad \text{Eq. 6.91} \Rightarrow \mathbf{S}_d\cdot\mathbf{S}_e = \frac{1}{2}(S^2 - S_e^2 - S_d^2).$$

Electron has spin $\frac{1}{2}$, so $S_e^2 = \frac{1}{2}\left(\frac{3}{2}\right)\hbar^2 = \frac{3}{4}\hbar^2$; deuteron has spin 1, so $S_d^2 = 1(2)\hbar^2 = 2\hbar^2$.
Total spin could be $\frac{3}{2}$ [in which case $S^2 = \frac{3}{2}\left(\frac{5}{2}\right)\hbar^2 = \frac{15}{4}\hbar^2$] or $\frac{1}{2}$ [in which case $S^2 = \frac{3}{4}\hbar^2$]. Thus

$$\langle \mathbf{S}_d\cdot\mathbf{S}_e\rangle = \left\{\begin{array}{l} \frac{1}{2}\left(\frac{15}{4}\hbar^2 - \frac{3}{4}\hbar^2 - 2\hbar^2\right) = \frac{1}{2}\hbar^2 \\ \frac{1}{2}\left(\frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2 - 2\hbar^2\right) = -\hbar^2 \end{array}\right\};\ \text{the difference is } \frac{3}{2}\hbar^2,\ \text{so}\quad \Delta E = \frac{\mu_0 g_d e^2\hbar^2}{2\pi m_d m_e a^3}.$$

But $\mu_0\epsilon_0 = \frac{1}{c^2} \Rightarrow \mu_0 = \frac{1}{\epsilon_0 c^2}$, so $\Delta E = \frac{2g_d e^2\hbar^2}{4\pi\epsilon_0 m_d m_e c^2 a^3} = \frac{2g_d\hbar^4}{m_d m_e^2 c^2 a^4} = \frac{3}{2}\frac{g_d}{g_p}\frac{m_p}{m_d}\Delta E_{\text{hydrogen}}$ (Eq. 6.98).
Now, $\lambda = \frac{c}{\nu} = \frac{ch}{\Delta E}$, so $\lambda_d = \frac{2}{3}\frac{g_p}{g_d}\frac{m_d}{m_p}\lambda_h$, and since $m_d = 2m_p$, $\lambda_d = \frac{4}{3}\left(\frac{5.59}{1.71}\right)(21\text{ cm}) = \boxed{92\text{ cm.}}$

Problem 6.39

(a) The potential energy of the electron (charge $-e$) at (x, y, z) due to q's at $x = \pm d$ alone is:

$$V = -\frac{eq}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x+d)^2+y^2+z^2}} + \frac{1}{\sqrt{(x-d)^2+y^2+z^2}}\right].\quad \text{Expanding (with } d \gg x, y, z):$$