Phys 402: Applications of Quantum Mechanics

Homework IV (due 930am, Thursday, Feb 9, 2012)

[To receive full credits, please show all necessary steps that lead to your answers.]

1) Problem 6.16 (only a), c), d) and f)).

2) During our lectures, we have introduce the raising and lowering operators for angular momentum and spins. In Prob. 4.18 (page 166), you need to further find out their actions.

or, tacking on the normalization factor,

$$\psi n00 = \frac{1}{\sqrt{\pi}\,(na)^{3/2}}e^{-r/na}, \quad \langle\psi_{n00}|p^4\psi_{m00}\rangle = \frac{8\hbar^4}{a^4}\frac{(n-m)}{(nm)^{5/2}} + \langle p^4\psi_{n00}|\psi_{m00}\rangle,$$

and hence p^4 is not Hermitian, for such states.

Problem 6.16

(a)

$$[\mathbf{L}\cdot\mathbf{S}, L_x] = [L_xS_x + L_yS_y + L_zS_z, L_x] = S_x\,[L_x, L_x] + S_y\,[L_y, L_x] + S_z\,[L_z, L_x]$$

$$= S_x(0) + S_y(-i\hbar L_z) + S_z(i\hbar L_y) = i\hbar(L_yS_z - L_zS_y) = i\hbar(\mathbf{L}\times\mathbf{S})_x.$$

Same goes for the other two components, so $\boxed{[\mathbf{L}\cdot\mathbf{S}, \mathbf{L}] = i\hbar(\mathbf{L}\times\mathbf{S}).}$

(b) $[\mathbf{L}\cdot\mathbf{S}, \mathbf{S}]$ is identical, only with $\mathbf{L}\leftrightarrow\mathbf{S}$: $\boxed{[\mathbf{L}\cdot\mathbf{S}, \mathbf{S}] = i\hbar(\mathbf{S}\times\mathbf{L}).}$

(c) $[\mathbf{L}\cdot\mathbf{S}, \mathbf{J}] = [\mathbf{L}\cdot\mathbf{S}, \mathbf{L}] + [\mathbf{L}\cdot\mathbf{S}, \mathbf{S}] = i\hbar(\mathbf{L}\times\mathbf{S} + \mathbf{S}\times\mathbf{L}) = \boxed{0.}$

(d) L^2 commutes with all components of $\mathbf{L}$ (and $\mathbf{S}$) , so $\boxed{[\mathbf{L}\cdot\mathbf{S}, L^2] = 0.}$

(e) Likewise, $\boxed{[\mathbf{L}\cdot\mathbf{S}, S^2] = 0.}$

(f) $[\mathbf{L}\cdot\mathbf{S}, J^2] = [\mathbf{L}\cdot\mathbf{S}, L^2] + [\mathbf{L}\cdot\mathbf{S}, S^2] + 2\,[\mathbf{L}\cdot\mathbf{S}, \mathbf{L}\cdot\mathbf{S}] = 0+0+0 \Longrightarrow \boxed{[\mathbf{L}\cdot\mathbf{S}, J^2] = 0.}$

Problem 6.17

With the plus sign, $j = l + 1/2$ $(l = j - 1/2)$: Eq. 6.57 $\Rightarrow E_r^1 = -\frac{(E_n)^2}{2mc^2}\left(\frac{4n}{j} - 3\right).$

$$\text{Equation 6.65} \Rightarrow E_{\text{so}}^1 = \frac{(E_n)^2}{mc^2}\frac{n\left[j(j+1) - (j-\frac{1}{2})(j+\frac{1}{2}) - \frac{3}{4}\right]}{(j-\frac{1}{2})j(j+\frac{1}{2})}$$
$$= \frac{(E_n)^2}{mc^2}\frac{n(j^2+j-j^2+\frac{1}{4}-\frac{3}{4})}{(j-\frac{1}{2})j(j+\frac{1}{2})} = \frac{(E_n)^2}{mc^2}\frac{n}{j(j+\frac{1}{2})}.$$

$$E_{\text{fs}}^1 = E_r^1 + E_{\text{so}}^1 = \frac{(E_n)^2}{2mc^2}\left(-\frac{4n}{j} + 3 + \frac{2n}{j(j+\frac{1}{2})}\right)$$

$$= \frac{(E_n)^2}{2mc^2}\left\{3 + \frac{2n}{j(j+\frac{1}{2})}\left[1 - 2\left(j + \frac{1}{2}\right)\right]\right\} = \frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j+\frac{1}{2}}\right).$$

Prob. 4.18

Sol:

$$\langle f_\ell^m | L_- L_+ | f_\ell^m \rangle = \langle f_\ell^m | L^2 - L_z^2 - \hbar L_z | f_\ell^m \rangle$$

$$= \ell(\ell+1) - (m+1)m$$

$$= \langle L_+ f_\ell^m | L_+ f_\ell^m \rangle = |A_\ell^m|^2$$

So:

$$A_\ell^m = \sqrt{\ell(\ell+1) - (m+1)m}.$$

Note

$$L^2 = L_z^2 + L_x^2 + L_y^2$$

$$= L_z^2 + \frac{1}{2}(L_+ L_- + L_- L_+)$$

and

$$[L_+, L_-] = 2\hbar L_z$$

So $L^2 = L_z^2 + \hbar L_z + L_- L_+$

or $\boxed{L_- L_+ = L^2 - L_z^2 - \hbar L_z}$

Some students might construct states using $J^2 = (S+L)^2$,

$J_z = S_z + L_z$. $|3/2, j_z\rangle$, $j_z = \pm 3/2, \pm 1/2$; $|1/2, j_z\rangle$, $j_z = \pm 1/2$.

3) For S=1/2 electrons with l=1, find all the eigen states and eigen values for the spin-orbit Hamiltonian. Hint: there are six states which are degenerate when the spin-orbit coupling is absent. Find the Hamiltonian matrix of spin-orbit coupling for the 6-fold degenerate states. Apply the degenerate perturbation theory. Only very few matrix elements are nonzero.

Sol: $H_{so} = A\vec{S}\cdot\vec{L} = A\left(S_z L_z + \frac{1}{2}(S_+ L_- + S_- L_+)\right)$

$j_z = 3/2$ $1/2$ $-1/2$ $3/2$

$|+\rangle\otimes|1,1\rangle$ $|-\rangle\otimes|1,1\rangle$ $|+\rangle\otimes|1,0\rangle$ $|-\rangle\otimes|1,0\rangle$ $|+\rangle\otimes|1,-1\rangle$ $|-\rangle\otimes|1,-1\rangle$

Hamiltonian Matrix

$$\begin{pmatrix} \frac{1}{2} & & & & & \\ & -\frac{1}{2}-E & \frac{1}{\sqrt{2}} & & & \\ & \frac{1}{\sqrt{2}} & -E & & & \\ & & & -E & \frac{1}{\sqrt{2}} & \\ & & & \frac{1}{\sqrt{2}} & -\frac{1}{2}-E & \\ & & & & & \frac{1}{2} \end{pmatrix}$$

$= \dfrac{H_{so}}{A} - E\hat{I}$.

$E_1 = \dfrac{A}{2}$, 4-fold deg.

$$\begin{cases} |+\rangle\otimes|1,1\rangle = \left|\frac{3}{2}, \frac{3}{2}\right\rangle \\ \frac{1}{\sqrt{3}}\left(|-\rangle\otimes|1,1\rangle + \sqrt{2}\,|+\rangle\otimes|1,0\rangle\right) = \left|\frac{3}{2}, \frac{1}{2}\right\rangle \\ \frac{1}{\sqrt{3}}\left(\sqrt{2}\,|-\rangle\otimes|1,0\rangle + |+\rangle\otimes|1,-1\rangle\right) = \left|\frac{3}{2}, -\frac{1}{2}\right\rangle \\ |-\rangle\otimes|1,-1\rangle = \left|\frac{3}{2}, -\frac{3}{2}\right\rangle \end{cases}$$

$E_2 = -A$, 2-fold deg.

$$\begin{cases} \frac{1}{\sqrt{3}}\left(-\sqrt{2}\,|-\rangle\otimes|1,1\rangle + |+\rangle\otimes|1,0\rangle\right) = \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \\ \frac{1}{\sqrt{3}}\left(-|-\rangle\otimes|1,0\rangle + \sqrt{2}\,|+\rangle\otimes|1,-1\rangle\right) = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \end{cases}$$

Alternatively,

$$[J^2, H_{so}] = [J_z, H_{so}] = 0$$

$$H_{so} = \frac{A}{2}(J^2 - S^2 - L^2)$$

$|+\rangle \otimes |1,1\rangle = |\frac{3}{2}, \frac{3}{2}\rangle$ One can show $J_+|\frac{3}{2},\frac{3}{2}\rangle = 0$

$\qquad\qquad\qquad\quad \uparrow \;\; \uparrow$

$\qquad\qquad\qquad\quad j \;\; j_z$

and $J^2|\frac{3}{2},\frac{3}{2}\rangle = \frac{3}{2}\times(\frac{3}{2}+1)|\frac{3}{2},\frac{3}{2}\rangle$

Using the lowering operator, one can produce the rest of states with $j = 3/2$: $|\frac{3}{2},\frac{1}{2}\rangle$, $|\frac{3}{2},-\frac{1}{2}\rangle$, $|\frac{3}{2},-\frac{3}{2}\rangle$.

Further more $|\frac{1}{2},\frac{1}{2}\rangle$, $|\frac{1}{2},-\frac{1}{2}\rangle$ for $j = \frac{1}{2}$.

For $j = \frac{3}{2}$, $E_1 = \frac{A}{2}\left(\frac{3}{2}\times\frac{5}{2} - \frac{1}{2}\times\frac{3}{2} - 2\right) = \frac{A}{2}$

4-fold degenerate

$j = 1/2$, $E_2 = \frac{A}{2}\left(\frac{1}{2}\times\frac{3}{2} - \frac{1}{2}\times\frac{3}{2} - 2\right) = -A$

2-fold degenerate

The results are identical to the 1st approach.

Note to get full credits, students need express $|j, j_z\rangle$ states in terms of $|s, s_z\rangle \otimes |l, l_z\rangle$ explicitly.