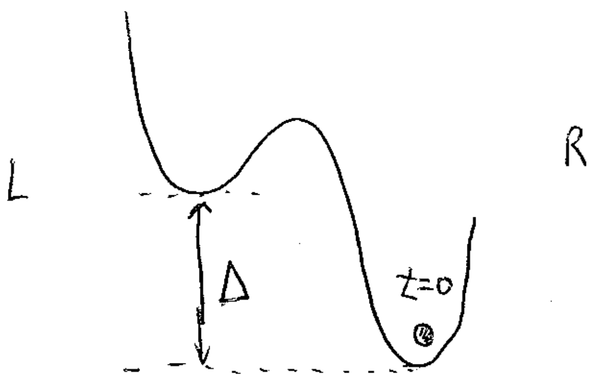
Phys 402: Applications of Quantum Mechanics

Homework III (due 930am, Thursday, Feb 2, 2012)

Quantum Bouncing in a double-well potential

A tilted double-well. Assume a particle initially is at the right hand side or the lower site of potential. Can the particle appear at the left hand site of the potential at time t? What is the probability? What happens when the tilt is very large and very small? Use a reasonable model to describe the physics phenomenon here. (Δ = tilt)



Sol:

$$H' = \begin{pmatrix} \frac{\Delta}{2} & -t \\ -t & -\frac{\Delta}{2} \end{pmatrix}$$

($|L\rangle - |R\rangle$ are the base vectors)

The particle will appear in the LHS of the potential due to tunneling.

$$\psi(t=0) = |R\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2\mathcal{E}_0}\left[\begin{pmatrix} t \\ \frac{\Delta}{2}+\mathcal{E}_0 \end{pmatrix} e^{-i\frac{\mathcal{E}_0 t}{\hbar}} - \begin{pmatrix} t \\ \frac{\Delta}{2}-\mathcal{E}_0 \end{pmatrix} e^{i\frac{\mathcal{E}_0 t}{\hbar}}\right]$$

$\mathcal{E}_0 = \sqrt{\left(\frac{\Delta}{2}\right)^2 + t^2}$, two eigen values are $E_{1,2} = \pm\sqrt{\left(\frac{\Delta}{2}\right)^2+t^2} = \pm\mathcal{E}_0$.

$$\rho_L(t=T) = \left(\frac{t}{\mathcal{E}_0}\right)^2 \sin^2\frac{\mathcal{E}_0 t}{\hbar}.$$

when $\Delta \ll t$, $\rho_L(t=T) \approx \sin^2\left(\frac{\mathcal{E}_0 t}{\hbar}\right)$;

$\Delta \gg t$, $\rho_L(t=T) = \left(\frac{2t}{\Delta}\right)^2 \sin^2\left(\frac{\mathcal{E}_0 t}{\hbar}\right)$; Amplitude $\sim \frac{4t^2}{\Delta^2}$

Classical limit. $t \to 0$, $\rho_L(t=T) \to 0$

Spin-1/2 electron

$$S_\alpha = \frac{\hbar}{2}\sigma_\alpha, \qquad \sigma_x = \begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1) Find the eigenspinors and eigenvalues of spin projection operators along x, y, z direction. (i.e. S_α, $\alpha = x, y, z$.)

Sol:

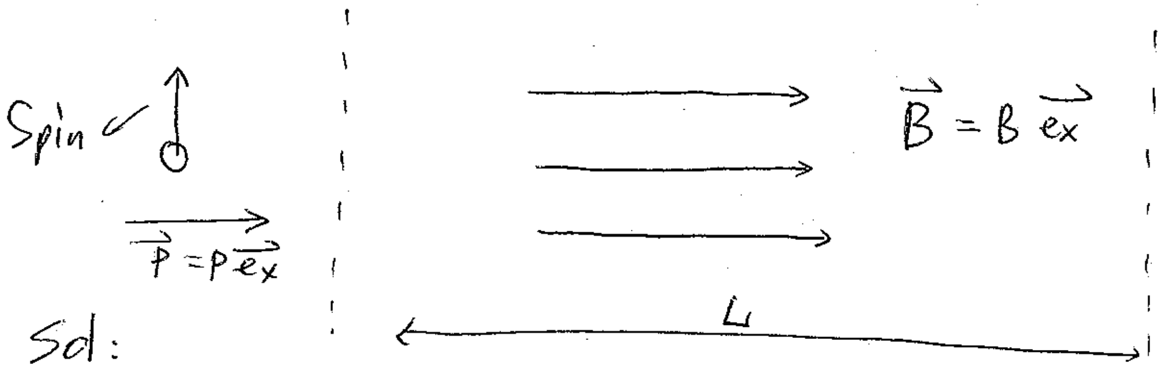
$$S_z \underset{|u\rangle}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = +\frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \qquad |u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

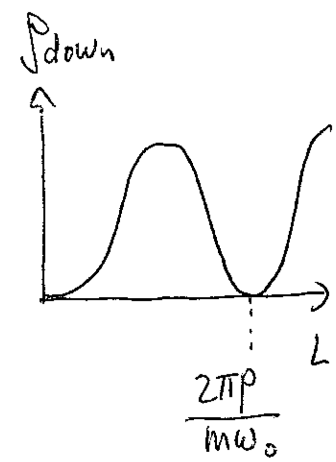
For S_x, $\quad |u\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |d\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

For S_y, $\quad |u\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad |d\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

2) Prob. 4.30 (page 178).

3) A polarized electron beam with spins pointing along the z-direction and momentum p enters a region with uniform magnetic fields pointing along the direction of x. Find out the outgoing spin states and the probability of finding a spin pointing along the downward direction. Assume the width of the region is L. Plot it as a function of L.





Sol:

translational Motion treated Classically.

$$t = \frac{L}{p/m} = \frac{mL}{p}, \qquad S_z \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}.$$

$$\psi(x=0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix} 1 \\ +1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$

$$\psi(x=L) = \frac{1}{2}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega_0 t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\omega_0 t}\right], \qquad \omega_0 = \frac{eB}{2m}$$

$$= \begin{pmatrix} \cos\omega_0 t \\ -i\sin\omega_0 t \end{pmatrix}, \qquad \boxed{\rho_{down} = \sin^2\left(\frac{\omega_0 m L}{p}\right).}$$

(c) $\boxed{\frac{\hbar^2}{4}, \text{ with probability 1.}}$

Problem 4.30

$$\mathsf{S}_r = \mathbf{S}\cdot\hat{r} = \mathsf{S}_x \sin\theta\cos\phi + \mathsf{S}_y \sin\theta\sin\phi + \mathsf{S}_z\cos\theta$$
$$= \frac{\hbar}{2}\left[\begin{pmatrix} 0 & \sin\theta\cos\phi \\ \sin\theta\cos\phi & 0\end{pmatrix} + \begin{pmatrix} 0 & -i\sin\theta\sin\phi \\ i\sin\theta\sin\phi & 0\end{pmatrix} + \begin{pmatrix}\cos\theta & 0 \\ 0 & -\cos\theta\end{pmatrix}\right]$$
$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta\end{pmatrix} = \boxed{\frac{\hbar}{2}\begin{pmatrix}\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta\end{pmatrix}.}$$

$$\begin{vmatrix} (\frac{\hbar}{2}\cos\theta - \lambda) & \frac{\hbar}{2}e^{-i\phi}\sin\theta \\ \frac{\hbar}{2}e^{i\phi}\sin\theta & (-\frac{\hbar}{2}\cos\theta - \lambda)\end{vmatrix} = -\frac{\hbar^2}{4}\cos^2\theta + \lambda^2 - \frac{\hbar^2}{4}\sin^2\theta = 0 \Rightarrow$$

$$\lambda^2 = \frac{\hbar^2}{4}(\sin^2\theta + \cos^2\theta) = \frac{\hbar^2}{4} \Rightarrow \boxed{\lambda = \pm\frac{\hbar}{2}} \text{ (of course).}$$

$$\frac{\hbar}{2}\begin{pmatrix}\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \Rightarrow \alpha\cos\theta + \beta e^{-i\phi}\sin\theta = \pm\alpha; \quad \beta = e^{i\phi}\frac{(\pm 1 - \cos\theta)}{\sin\theta}\alpha.$$

Upper sign: Use $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$, $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$. Then $\beta = e^{i\phi}\frac{\sin(\theta/2)}{\cos(\theta/2)}\alpha$. Normalizing:

$$1 = |\alpha|^2 + |\beta|^2 = |\alpha^2| + \frac{\sin^2(\theta/2)}{\cos^2(\theta/2)}|\alpha|^2 = |\alpha|^2\frac{1}{\cos^2(\theta/2)} \Rightarrow \alpha = \cos\frac{\theta}{2},\ \beta = e^{i\phi}\sin\frac{\theta}{2},\ \boxed{\chi_+^{(r)} = \begin{pmatrix}\cos(\theta/2) \\ e^{i\phi}\sin(\theta/2)\end{pmatrix}.}$$

Lower sign: Use $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$, $\beta = -e^{i\phi}\frac{\cos(\theta/2)}{\sin(\theta/2)}\alpha$; $1 = |\alpha|^2 + \frac{\cos^2(\theta/2)}{\sin^2(\theta/2)}|\alpha|^2 = |\alpha|^2\frac{1}{\sin^2(\theta/2)}$.

Pick $\alpha = e^{-i\phi}\sin(\theta/2)$; then $\beta = -\cos(\theta/2)$, and $\boxed{\chi_-^{(r)} = \begin{pmatrix} e^{-i\phi}\sin(\theta/2) \\ -\cos(\theta/2)\end{pmatrix}.}$

Problem 4.31

There are three states: $\chi_+ = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\chi_0 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\chi_- = \begin{pmatrix}0\\0\\1\end{pmatrix}$.

$\mathsf{S}_z\chi_+ = \hbar\chi_+$, $\mathsf{S}_z\chi_0 = 0$, $\mathsf{S}_z\chi_- = -\hbar\chi_-$, $\Rightarrow$ $\boxed{\mathsf{S}_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0 \\ 0 & 0 & -1\end{pmatrix}.}$ From Eq. 4.136:

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