Phys 402: Applications of Quantum Mechanics

Homework I (due 930am, Thursday, Jan 14, 2016)

1) Problem 6.2, page 254.

2) Problem 6.3. (Two identical Boson wave functions have to be symmetrized. See page 203-205.)

3) Problem 6.29, page 286.

4) Stark effect of a charged particle moving in a Harmonic potential. Now assume an electric field along X-direction is applied to the charged particle subject to a harmonic potential defined in Prob. 6.2. Find the second order correction to the ground state energy using the second order perturbation theory.

   Hint: A very pleasant approach to the harmonic potential problem is to use lowering/raising operators introduced in section 2.3. These operators are closely related to the creation-annihilation operators widely used in quantum optics and in quantum electrodynamics. Although for this particle problem this is not the only way to proceed.

5) Stark effect in a hydrogen atom. Calculate the second order correction to the ground state energy and polarizability. (You can keep the first 4 excited states in your estimate to simplify the calculation.)

* If integrals and sum can not be done exactly, please reduce them to dimensionless ones and show that they are convergent. Alternatively evaluate them numerically.

Problem 6.2

(a) $E_n = (n+\frac{1}{2})\hbar\omega'$, where $\omega' \equiv \sqrt{k(1+\epsilon)/m} = \omega\sqrt{1+\epsilon} = \omega(1+\frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \frac{1}{16}\epsilon^3\cdots)$, so

$$\boxed{E_n = (n+\tfrac{1}{2})\hbar\omega\sqrt{1+\epsilon} = (n+\tfrac{1}{2})\hbar\omega(1+\tfrac{1}{2}\epsilon - \tfrac{1}{8}\epsilon^2 + \cdots).}$$

(b) $H' = \frac{1}{2}k'x^2 - \frac{1}{2}kx^2 = \frac{1}{2}kx^2(1+\epsilon-1) = \epsilon(\frac{1}{2}kx^2) = \epsilon V$, where V is the unperturbed potential energy. So $E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle = \epsilon\langle n|V|n\rangle$, with $\langle n|V|n\rangle$ the expectation value of the (unperturbed) potential energy in the n^{th} unperturbed state. This is most easily obtained from the virial theorem (Problem 3.31), but it can also be derived algebraically. In this case the virial theorem says $\langle T\rangle = \langle V\rangle$. But $\langle T\rangle + \langle V\rangle = E_n$. So $\langle V\rangle = \frac{1}{2}E_n^0 = \frac{1}{2}(n+\frac{1}{2})\hbar\omega$; $\boxed{E_n^1 = \frac{\epsilon}{2}(n+\frac{1}{2})\hbar\omega,}$ which is precisely the ϵ^1 term in the power series from part (a).

Problem 6.3

(a) In terms of the one-particle states (Eq. 2.28) and energies (Eq. 2.27):

Ground state: $\psi_1^0(x_1,x_2) = \psi_1(x_1)\psi_1(x_2) = \boxed{\frac{2}{a}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right);}$ $E_1^0 = 2E_1 = \boxed{\frac{\pi^2\hbar^2}{ma^2}.}$

First excited state: $\psi_2^0(x_1,x_2) = \frac{1}{\sqrt{2}}\left[\psi_1(x_1)\psi_2(x_2) + \psi_2(x_1)\psi_1(x_2)\right]$

$$= \boxed{\frac{\sqrt{2}}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) + \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right];} \quad E_2^0 = E_1 + E_2 = \boxed{\frac{5}{2}\frac{\pi^2\hbar^2}{ma^2}.}$$

(b)

$$E_1^1 = \langle\psi_1^0|H'|\psi_1^0\rangle = (-aV_0)\left(\frac{2}{a}\right)^2\int_0^a\int_0^a \sin^2\left(\frac{\pi x_1}{a}\right)\sin^2\left(\frac{\pi x_2}{a}\right)\delta(x_2 - x_2)\,dx_1\,dx_2$$

$$= -\frac{4V_0}{a}\int_0^a \sin^4\left(\frac{\pi x}{a}\right)dx = -\frac{4V_0}{a}\frac{a}{\pi}\int_0^\pi \sin^4 y\,dy = -\frac{4V_0}{\pi}\cdot\frac{3\pi}{8} = \boxed{-\frac{3}{2}V_0.}$$

$$E_2^1 = \langle\psi_2^0|H'|\psi_2^0\rangle$$

$$= (-aV_0)\left(\frac{2}{a^2}\right)\iint_0^a\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) + \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right]^2\delta(x_1 - x_2)\,dx_1\,dx_2$$

$$= -\frac{2V_0}{a}\int_0^a\left[\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right) + \sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\right]^2 dx$$

$$= -\frac{8V_0}{a}\int_0^a \sin^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{2\pi x}{a}\right)dx = -\frac{8V_0}{a}\cdot\frac{a}{\pi}\int_0^\pi \sin^2 y\sin^2(2y)\,dy$$

$$= -\frac{8V_0}{\pi}\cdot 4\int_0^\pi \sin^2 y\sin^2 y\cos^2 y\,dy = -\frac{32V_0}{\pi}\int_0^\pi(\sin^4 y - \sin^6 y)\,dy$$

$$= -\frac{32V_0}{\pi}\left(\frac{3\pi}{8} - \frac{5\pi}{16}\right) = \boxed{-2V_0.}$$

The first angular integral is $3(4\pi/3)(\mathbf{S}_p \cdot \mathbf{S}_e) = 4\pi(\mathbf{S}_p \cdot \mathbf{S}_e)$, while the second is $-(\mathbf{S}_p \cdot \mathbf{S}_e)\int \sin\theta\, d\theta\, d\phi = -4\pi(\mathbf{S}_p \cdot \mathbf{S}_e)$, so the two cancel, and the result is zero. QED [Actually, there is a little sleight-of-hand here, since for $l = 0$, $\psi \to$ constant as $r \to 0$, and hence the radial integral diverges logarithmically at the origin. Technically, the first term in Eq. 6.86 is the field outside an infinitesimal *sphere*; the delta-function gives the field *inside*. For this reason it is correct to do the angular integral first (getting zero) and not worry about the radial integral.]

Problem 6.28

From Eq. 6.89 we see that $\Delta E \propto \left(\frac{g}{m_p m_e a^3}\right)$; we want reduced mass in a, but *not* in $m_p m_e$ (which come from Eq. 6.85); the notation in Eq. 6.93 obscures this point.

(a) g and m_p are unchanged; $m_e \to m_\mu = 207 m_e$, and $a \to a_\mu$. From Eq. 4.72, $a \propto 1/m$, so

$$\frac{a}{a_\mu} = \frac{m_\mu(\text{reduced})}{m_e} = \frac{m_\mu m_p}{m_\mu + m_p} \cdot \frac{1}{m_e} = \frac{207}{1 + 207(m_e/m_p)} = \frac{207}{1 + 207\frac{(9.11\times10^{-31})}{1.67\times10^{-27})}} = \frac{207}{1.11} = 186.$$

$$\Delta E = (5.88 \times 10^{-6}\text{ eV})\,(1/207)\,(186)^3 = \boxed{0.183\text{ eV.}}$$

(b) $g: 5.59 \to 2;\quad m_p \to m_e;\quad \frac{a}{a_p} = \frac{m_p(\text{reduced})}{m_e} = \frac{m_e^2}{m_e + m_e} \cdot \frac{1}{m_e} = \frac{1}{2}.$

$$\Delta E = (5.88 \times 10^{-6}\text{ eV})\left(\frac{2}{5.59}\right)\left(\frac{1.67 \times 10^{-27}}{9.11 \times 10^{-31}}\right)\left(\frac{1}{2}\right)^3 = \boxed{4.82 \times 10^{-4}\text{ eV.}}$$

(c) $g: 5.59 \to 2;\quad m_p \to m_\mu;\quad \frac{a}{a_m} = \frac{m_m(\text{reduced})}{m_e} = \frac{m_e m_\mu}{m_e + m_\mu} \cdot \frac{1}{m_e} = \frac{207}{208}.$

$$\Delta E = (5.88 \times 10^{-6})\left(\frac{2}{5.59}\right)\left(\frac{1.67 \times 10^{-27}}{(207)(9.11 \times 10^{-31})}\right)\left(\frac{207}{208}\right)^3 = \boxed{1.84\times10^{-5}\text{ eV.}}$$

Problem 6.29

Use perturbation theory:

$$H' = -\frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{b} - \frac{1}{r}\right), \quad \text{for} \quad 0 < r < b. \quad \Delta E = \langle\psi|H'|\psi\rangle, \quad \text{with} \quad \psi \equiv \frac{1}{\sqrt{\pi a^3}}e^{-r/a}.$$

$$\Delta E = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{\pi a^3}4\pi\int_0^b\left(\frac{1}{b} - \frac{1}{r}\right)e^{-2r/a}r^2\,dr = -\frac{e^2}{\pi\epsilon_0 a^3}\left(\frac{1}{b}\int_0^b r^2 e^{-2r/a}\,dr - \int_0^b r e^{-2r/a}\,dr\right)$$

$$= -\frac{e^2}{\pi\epsilon_0 a^3}\left\{\frac{1}{b}\left[-\frac{a}{2}r^2 e^{-2r/a} + a\left(\frac{a}{2}\right)^2 e^{-2r/a}\left(-\frac{2r}{a} - 1\right)\right] - \left[\left(\frac{a}{2}\right)^2 e^{-2r/a}\left(-\frac{2r}{a} - 1\right)\right]\right\}\Bigg|_0^b$$

$$= -\frac{e^2}{\pi\epsilon_0 a^3}\left[-\frac{a}{2b}b^2 e^{-2b/a} + \frac{a^3}{4b}e^{-2b/a}\left(-\frac{2b}{a} - 1\right) - \frac{a^2}{4}e^{-2b/a}\left(-\frac{2b}{a} - 1\right) + \frac{a^3}{4b} - \frac{a^2}{4}\right]$$

$$= -\frac{e^2}{\pi\epsilon_0 a^3}\left[e^{-2b/a}\left(-\frac{ab}{2} - \frac{a^2}{2} - \frac{a^3}{4b} + \frac{ab}{2} + \frac{a^2}{4}\right) + \frac{a^2}{4}\left(\frac{a}{b} - 1\right)\right]$$

$$= -\frac{e^2}{\pi\epsilon_0 a^3}\left[e^{-2b/a}\left(-\frac{a^2}{4}\right)\left(\frac{a}{b} + 1\right) + \frac{a^2}{4}\left(\frac{a}{b} - 1\right)\right] = \frac{e^2}{4\pi\epsilon_0 a}\left[\left(1 - \frac{a}{b}\right) + \left(1 + \frac{a}{b}\right)e^{-2b/a}\right].$$

Let $+2b/a = \epsilon$ (very small). Then the term in square brackets is:

$$\left(1-\frac{2}{\epsilon}\right)+\left(1+\frac{2}{\epsilon}\right)\left(1-\epsilon+\frac{\epsilon^2}{2}-\frac{\epsilon^3}{6}+\cdots\right)$$

$$=\cancel{1}-\frac{\cancel{2}}{\cancel{\epsilon}}+\cancel{1}+\frac{\cancel{2}}{\cancel{\epsilon}}-\cancel{\epsilon}-\cancel{2}+\frac{\epsilon^2}{2}+\cancel{\epsilon}-\frac{\epsilon^3}{6}-\frac{\epsilon^2}{3}+(\,)\epsilon^3+\cdots=\frac{\epsilon^2}{6}+(\,)\epsilon^3+(\,)\epsilon^4\cdots$$

To leading order, then, $\Delta E = \dfrac{e^2}{4\pi\epsilon_0}\dfrac{1}{a}\dfrac{4b^2}{6a^2}$.

$$E = E_1 = -\frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2}; \quad a = \frac{4\pi\epsilon_0\hbar^2}{me^2}; \quad \text{so} \quad Ea = -\frac{e^2}{2(4\pi\epsilon_0)}.$$

$$\frac{\Delta E}{E} = \frac{e^2}{4\pi\epsilon_0}\left(-\frac{2(4\pi\epsilon_0)}{e^2}\right)\frac{2b^2}{3a^2} = \boxed{-\frac{4}{3}\left(\frac{b}{a}\right)^2.}$$

Putting in $a = 5\times 10^{-11}$ m:

$$\frac{\Delta E}{E} = -\frac{4}{3}\left(\frac{10^{-15}}{5\times 10^{-11}}\right) = -\frac{16}{3}\times 10^{-10} \approx \boxed{-5\times 10^{-10}.}$$

By contrast, $\begin{cases} \text{fine structure:} & \Delta E/E \approx \alpha^2 = (1/137)^2 = 5\times 10^{-5}, \\ \text{hyperfine structure:} & \Delta E/E \approx (m_e/m_p)\alpha^2 = (1/1800)(1/137)^2 = 3\times 10^{-8}. \end{cases}$

So the correction for the finite size of the nucleus is *much* smaller (about 1% of hyperfine).

Problem 6.30

(a) In terms of the one-dimensional harmonic oscillator states $\{\psi_n(x)\}$, the unperturbed ground state is

$$|0\rangle = \psi_0(x)\psi_0(y)\psi_0(z).$$

$$E_0^1 = \langle 0|H'|0\rangle = \langle\psi_0(x)\psi_0(y)\psi_0(z)|\lambda x^2yz|\psi_0(x)\psi_0(y)\psi_0(z)\rangle = \lambda\langle x^2\rangle_0\langle y\rangle_0\langle z\rangle_0.$$

But $\langle y\rangle_0 = \langle z\rangle_0 = 0$. So there is *no* change, in first order.

(b) The (triply degenerate) first excited states are

$$\begin{cases} |1\rangle = \psi_0(x)\psi_0(y)\psi_1(z) \\ |2\rangle = \psi_0(x)\psi_1(y)\psi_0(z) \\ |3\rangle = \psi_1(x)\psi_0(y)\psi_0(z) \end{cases}$$

In this basis the perturbation matrix is $W_{ij} = \langle i|H'|j\rangle, \quad i = 1,2,3.$

$$\langle 1|H'|1\rangle = \langle\psi_0(x)\psi_0(y)\psi_1(z)|\lambda x^2yz|\psi_0(x)\psi_0(y)\psi_1(z)\rangle = \lambda\langle x^2\rangle_0\langle y\rangle_0\langle z\rangle_1 = 0,$$

$$\langle 2|H'|2\rangle = \langle\psi_0(x)\psi_1(y)\psi_0(z)|\lambda x^2yz|\psi_0(x)\psi_1(y)\psi_0(z)\rangle = \lambda\langle x^2\rangle_0\langle y\rangle_1\langle z\rangle_0 = 0,$$

$$\langle 3|H'|3\rangle = \langle\psi_1(x)\psi_0(y)\psi_0(z)|\lambda x^2yz|\psi_1(x)\psi_0(y)\psi_0(z)\rangle = \lambda\langle x^2\rangle_1\langle y\rangle_0\langle z\rangle_0 = 0,$$

$$\langle 1|H'|2\rangle = \langle\psi_0(x)\psi_0(y)\psi_1(z)|\lambda x^2yz|\psi_0(x)\psi_1(y)\psi_0(z)\rangle = \lambda\langle x^2\rangle_0\langle 0|y|1\rangle\langle 1|z|0\rangle$$

$$= \lambda\frac{\hbar}{2m\omega}|\langle 0|x|1\rangle|^2 = \lambda\left(\frac{\hbar}{2m\omega}\right)^2 \quad \text{[using Problems 2.11 and 3.33].}$$

$$\langle 1|H'|3\rangle = \langle\psi_0(x)\psi_0(y)\psi_1(z)|\lambda x^2yz|\psi_1(x)\psi_0(y)\psi_0(z)\rangle = \lambda\langle 0|x^2|1\rangle\langle y\rangle_0\langle 1|z|0\rangle = 0,$$

$$\langle 2|H'|3\rangle = \langle\psi_0(x)\psi_1(y)\psi_0(z)|\lambda x^2yz|\psi_1(x)\psi_0(y)\psi_0(z)\rangle = \lambda\langle 0|x^2|1\rangle\langle 1|y|0\rangle_0\langle z\rangle_0 = 0.$$

Prob. 4).

Harmonic Oscillator state $|m\rangle^0$.

$$H' = -QEX, \qquad {}^0\langle m|H'|0\rangle^0 = -QE\sqrt{\frac{\hbar}{2m\omega}}\,\delta_{m,1}$$

$$E_n^1 = 0, \qquad E_n^2 = -\frac{(QE)^2}{2m\omega_0^2} = -\frac{1}{2}\alpha E^2,$$

$$\alpha = \frac{Q^2}{m\omega_0^2} = \frac{Q^2}{k} \leftarrow \text{Spring Const.}$$

Prob. 5). Only non-vanishing matrix element is (for n=2 states)

$$\langle n,l,m|H'|1,0,0\rangle = \delta_{n,2}\delta_{l,1}\delta_{m,0}\,[-ea_B G]$$

a_B - Bohr Radius. $\quad G = \frac{1}{\sqrt{3}}\int \frac{r}{a} R_{10}(r)R_{21}\, r^2 dr$

$$= \frac{1}{\sqrt{3}}\underbrace{\int \rho f(\rho)d\rho}$$

dimensionless integral order of One.

$$E_n^2 = -\frac{4e^2a_B^2G^2}{3E_0}E^2, \qquad \alpha = \frac{8}{3}\frac{e^2a_B^2}{E_0}G^2 \propto \varepsilon_0 a_B^3$$

$\uparrow$ vol. of an atom.

E_0 - Bohr binding energy

$\alpha \propto$ Vol of atoms