Phys529B: Topics of Quantum Theory

Lecture 15: identifying SIFP via Callan Symanzik RGE II: more applications

instructor: Fei Zhou

Λ_{UV} $\quad \phi_0(x) \qquad H(g_{01}, g_{02}, \dots; m, \Lambda_{UV})$

Λ $\quad \phi(x) \qquad H(g_1(\Lambda), g_2(\Lambda), \dots; m(\Lambda), \Lambda; Z(\Lambda))$

$\phi(x) = Z^{-1/2} \phi_R(x)$

or $\quad = Z^{1/2} \phi_0(x)$

Λ_{2R} $\quad \phi_R(x) \qquad H(g_{1R}, g_{2R}, \dots; m_R)$

Supplementary stuff on RGE

$$H = H(m, \lambda, \ldots ; \Lambda) = H(\tilde{m}, \tilde{\lambda}, \cdots, Z(\Lambda) ; \Lambda)$$

for any given Λ, there shall be a $\tilde{m}(\Lambda)$, $\tilde{\lambda}(\Lambda)$ ...

so that H leads to the same physics.

$$\frac{d\tilde{m}}{dt} = \beta_m(\tilde{m}, \tilde{\lambda}, \ldots) \qquad \frac{dZ(\Lambda)}{dt} = \gamma(\tilde{m}, \tilde{\lambda}, Z)$$

$$\frac{d\tilde{\lambda}}{dt} = \beta_{\tilde{\lambda}}(\tilde{m}, \tilde{\lambda}, \ldots) \qquad t = \ln\frac{\Lambda}{\Lambda_{uv}}$$

Renormalization Group Equations

Running scale

- Callan-Symanzik approach (also Coleman-Weinberg applications) : Utilizing Green’s functions to obtain RGEs and understand scale Transformations

- An interesting application to cold gases can be found in Yang, Jiang and Zhou, PRL 124 (22), 225701, 2020, Tricritical physics in 2D Superfluids; Jiang and Zhou, ArXiv:2301.12657 at https://arxiv.org/pdf/2301.12657.pdf, Interacting 3D Fermions in Planckian limit: Entropy and dynamics.

Approach 1: $\quad \phi(x) = Z^{1/2}(\Lambda)\,\phi_R(x)$

IPR

$$\langle 0|T\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle = G^{(n)}(x_1,\ldots x_n)$$

$$G^{(n)}(x_1,\ldots x_n) = Z^{n/2}(\Lambda)\,\underbrace{\langle 0|T\phi_R(x_1)\cdots\phi_R(x_n)|0\rangle_c}_{\text{independent of "}\Lambda\text{"}}.$$

Computed with $H(g_1(\Lambda), g_2(\Lambda), \cdots g_n(\Lambda); m(\Lambda), \Lambda)$

$$G^{(n)} = G^{(n)}(\ldots;\ \tilde{g}_1, \tilde{g}_2, \ldots \tilde{g}_n(\Lambda); \tilde{m}(\Lambda), t)$$

$$t = \ln\Lambda$$

$$\left\{ \frac{\partial}{\partial t} + \beta(g) \frac{\partial}{\partial g} \right\} G^{(n)} = \left\{ \underbrace{\frac{n}{2}\, \Lambda \frac{\delta Z}{\delta \Lambda} \frac{1}{Z}}_{\gamma(\tilde{g})} \right\} G^{(n)}$$

$$\left\{ \frac{\partial}{\partial t} + \beta(g) \frac{\partial}{\partial g} - \frac{n}{2} \gamma(g) \right\} G^{(n)} = 0$$

Note : $Z(t=0, \tilde{g}) = 1$

Approach II : $\phi(x) = Z^{-1/2}(\Lambda)\,\phi_0(x)$

$$\langle 0|\,T\,\phi(x_1)\,\phi(x_2)\cdots\phi(x_n)\,|0\rangle = G^{(n)}(x_1, \dots x_n)$$

$$G^{(n)}(x_1, \dots x_n) = Z^{-n/2}(\Lambda)\,\underbrace{\langle 0|\,T\,\phi_0(x_1)\cdots\phi_0\,x_2)\,|0\rangle}_{\text{independent of "}\Lambda\text{".}}{}_c$$

Computed with $H(g_1(\Lambda), g_2(\Lambda), \cdots g_n(\Lambda);\, m(\Lambda), \Lambda)$

$$G^{(n)} = G^{(n)}(\cdots;\ \tilde{g}_1, \tilde{g}_2, \dots \tilde{g}_n(\Lambda);\ \tilde{m}(\Lambda), t)$$

$$t = \ln \Lambda$$

$$\left\{ \frac{\partial}{\partial t} + \beta(g) \frac{\partial}{\partial g} \right\} G^{(n)} = \left\{ \frac{n}{2} \underbrace{\Lambda \frac{\delta Z}{\delta \Lambda} \frac{1}{Z}}_{\gamma(\tilde{g})} \right\} G^{(n)}$$

$$\left\{ \frac{\partial}{\partial t} + \beta(g) \frac{\partial}{\partial g} + \frac{n}{2} \gamma(\tilde{g}) \right\} G^{(n)} = 0$$

Note: $Z\left(t = \ln \frac{\Lambda_{uv}}{\Lambda_{IR}}, \tilde{g}\right) = 1$

$$\left\{\frac{\partial}{\partial t}+\beta(g)\frac{\partial}{\partial g}\right\}G^{(n)}=\left\{\underbrace{\frac{n}{2}\,\Lambda\frac{\delta Z}{\delta\Lambda}\,\frac{1}{Z}}_{\gamma(\tilde{g})}\right\}G^{(n)}$$

$$\left\{\frac{\partial}{\partial t}+\beta(g)\frac{\partial}{\partial g}-\gamma(\tilde{g})\right\}G^{(n)}=0$$

Note: $Z(t=0,\tilde{g})=1$, $\tilde{g}=g\Lambda^{d-3}$

$$\begin{cases} \gamma(\tilde{g}) = \dfrac{1}{Z}\dfrac{\partial Z}{\partial t} = 0 & (Z \text{ is from Two-loops) in QFT} \\ \gamma(\tilde{g}) = 0 & \text{in NRQFT.} \end{cases}$$

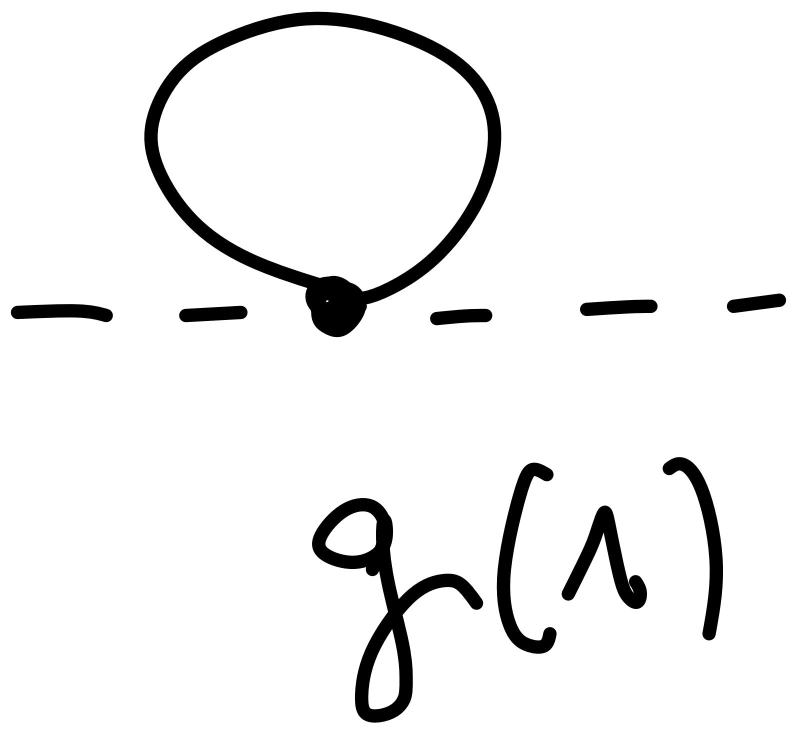
$$\rightarrow \beta(g)\frac{\partial}{\partial g}G^{(n)}(g) = -\frac{\partial}{\partial t}G^{(n)}(g)$$

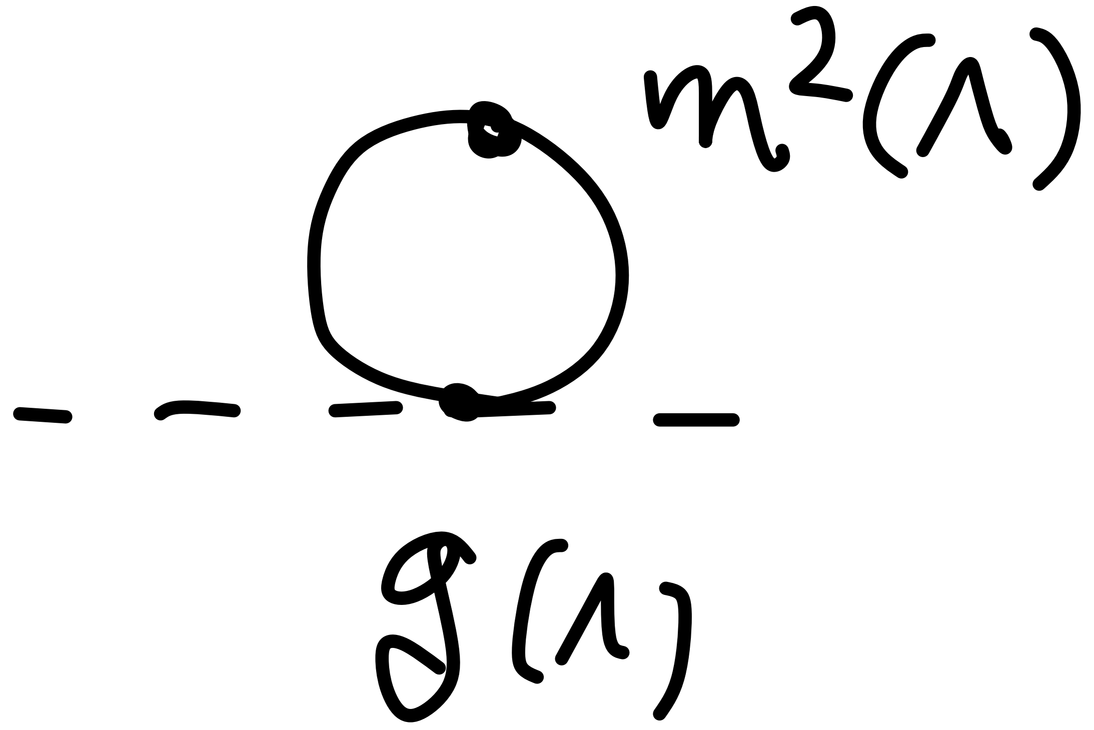
$G^{(2)}$

$\downarrow$

1PI

$=$



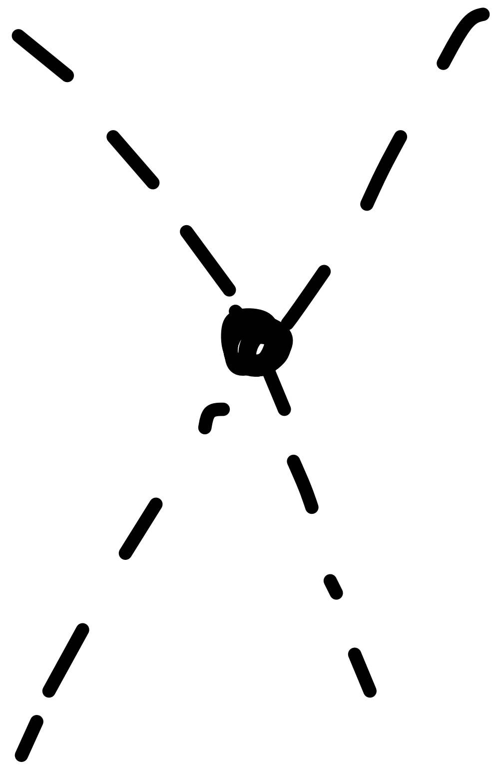


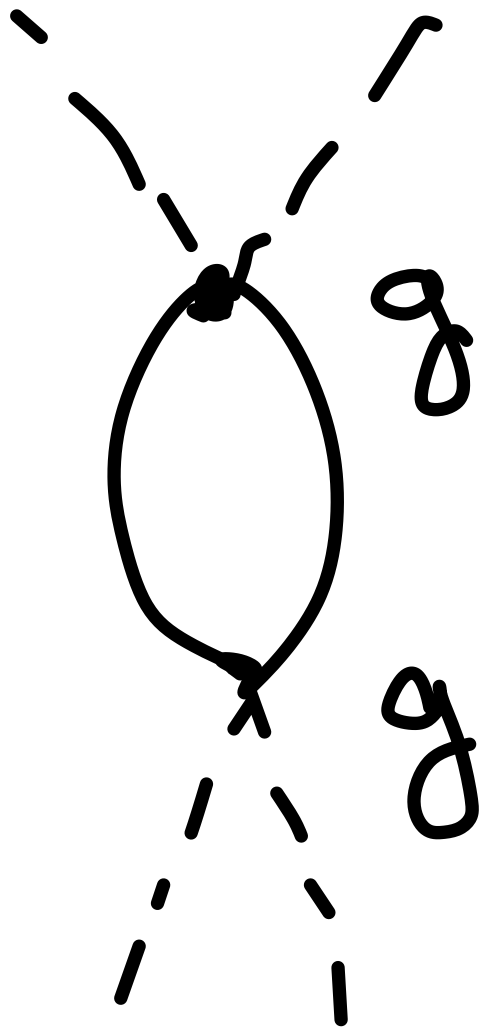
$G^{(4)}$

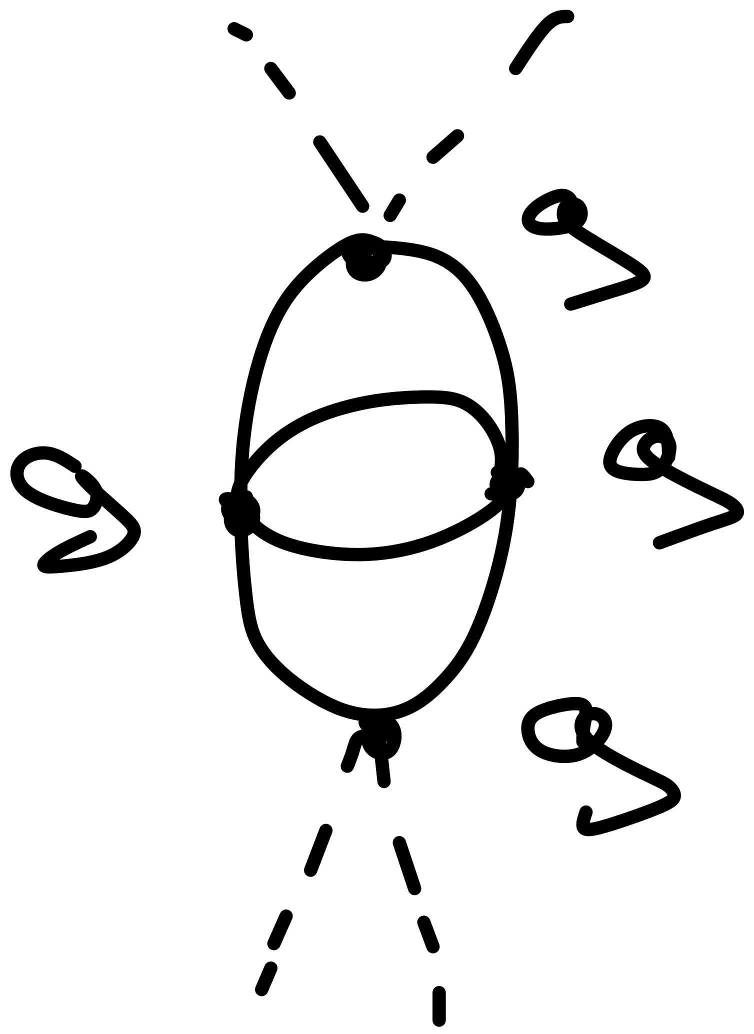
$\downarrow$

1PI

$=$



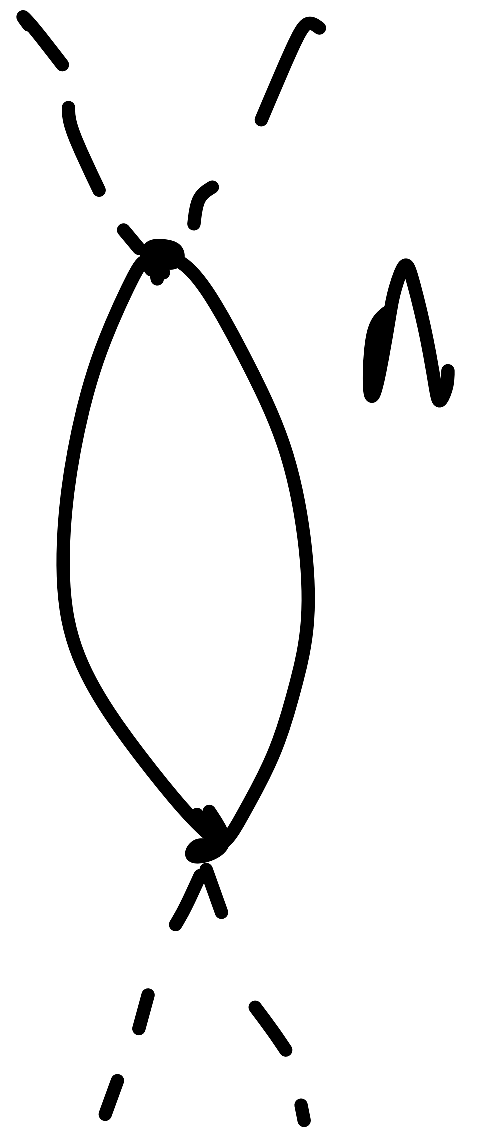




Lowest Order: $\tilde{g} = \Lambda^{d-3} g(\Lambda)$

$$\begin{cases} \beta(\tilde{g}) = (d-3)\,\tilde{g} + \beta(g)\,\Lambda^{d-3} \\ \beta(g) \doteq -\Lambda \dfrac{\partial}{\partial \Lambda}\, \delta g(\Lambda), \end{cases}$$

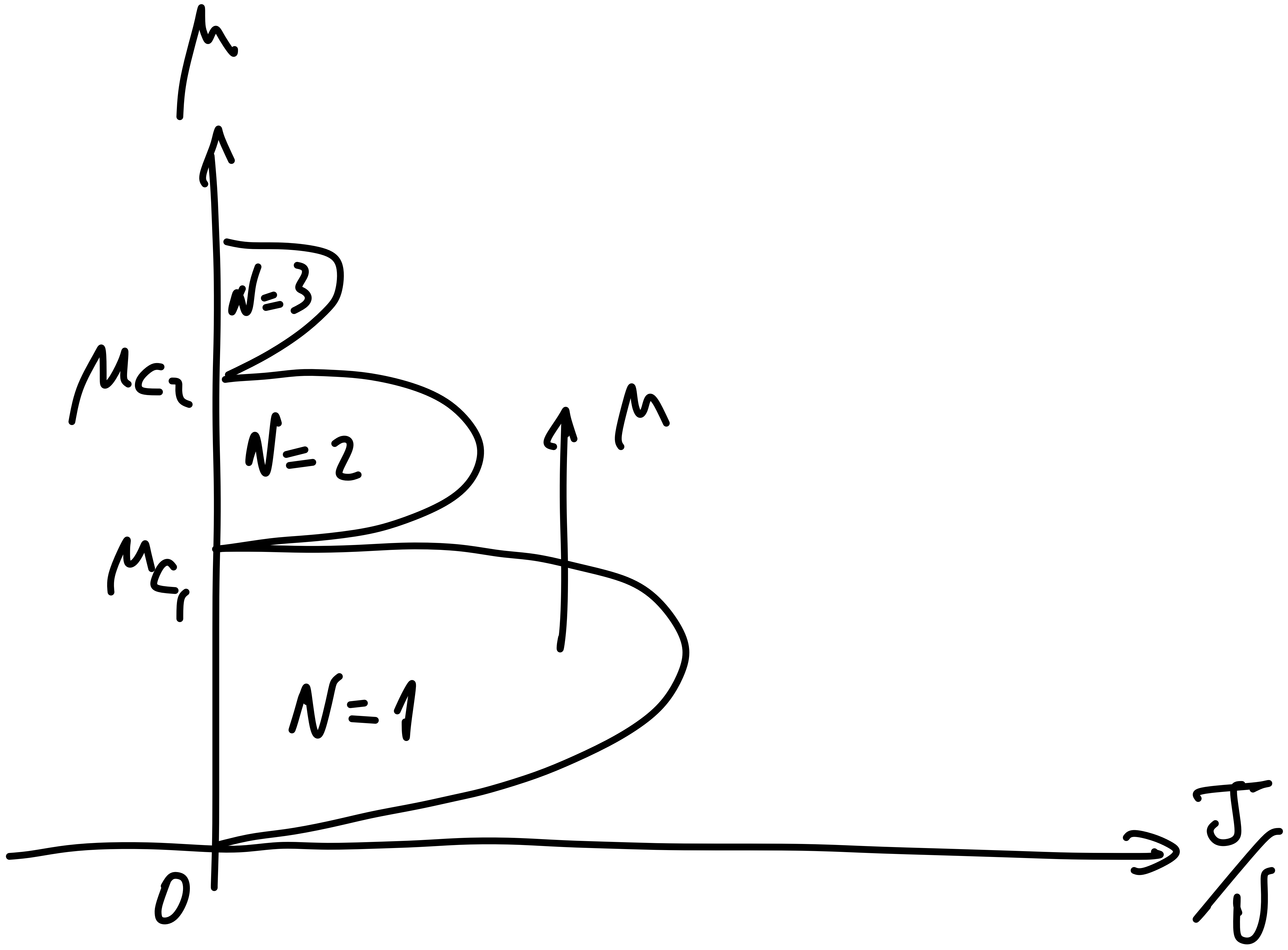
$\delta g(\Lambda) =$

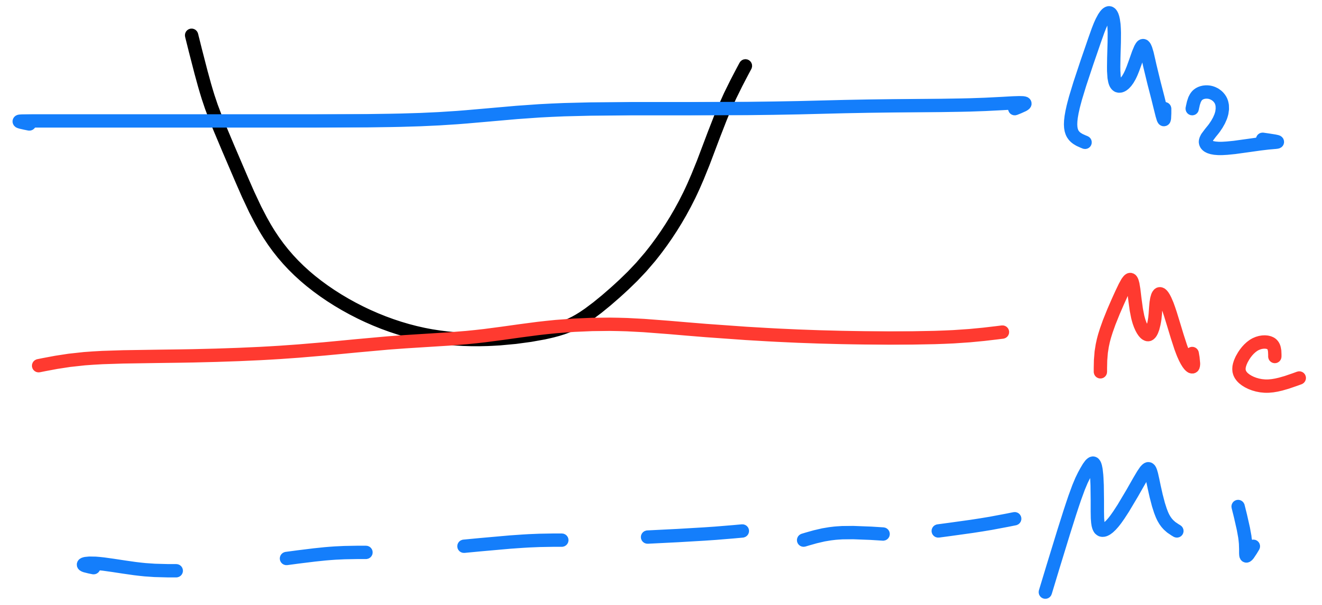


$$-g^2 \int^{\Lambda} \frac{1}{(\omega^2+p^2)^2} \frac{d\omega}{2\pi} \frac{d^{(d)}p}{(2\pi)^d}$$

$$= g^2 \Lambda^{d-3} \frac{\Omega_d}{(2\pi)^{d+1}}$$

$$\beta(\tilde{g}) = (d-3)\,\tilde{g} + \tilde{g}^2 c_d$$


μ
$N=3$
μ_{c_2}
$N=2$
μ
μ_{c_1}
$N=1$
0
J/U

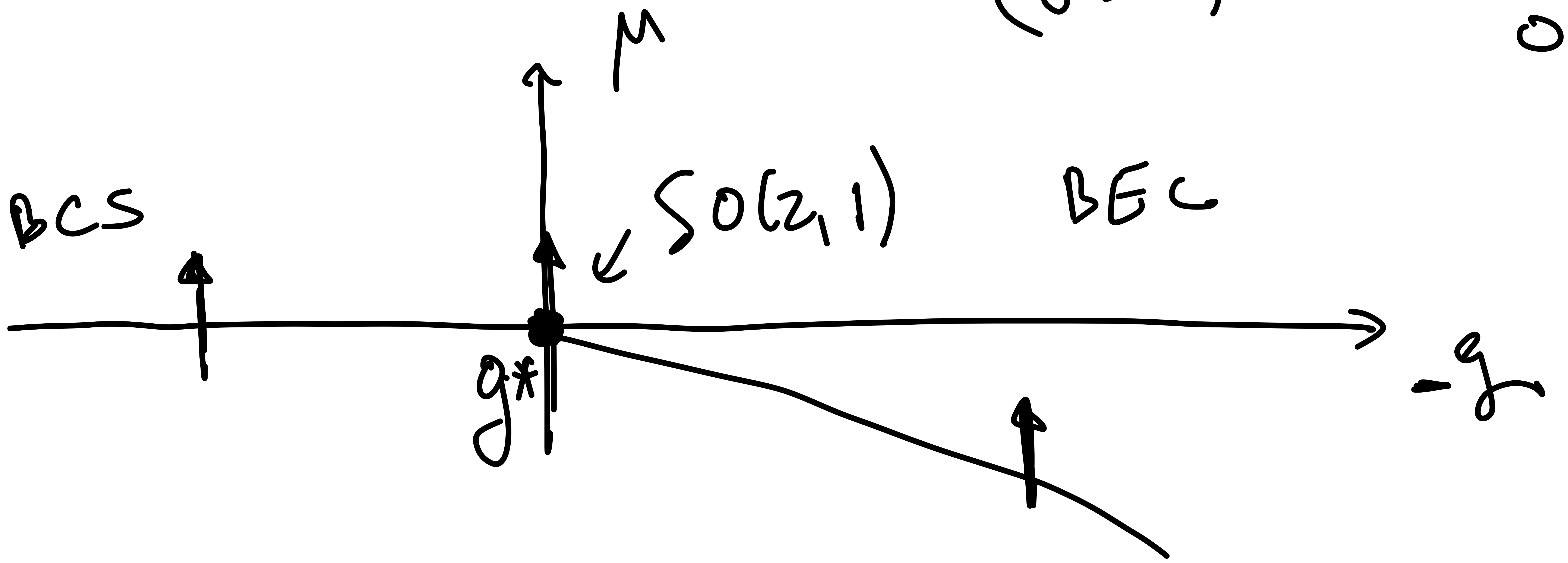

μ_2
μ_c
μ_1

NRQFT

$$\int \psi^\dagger (i\partial_t - H)\psi$$

$$\mathcal{L} = \int \psi^\dagger \underbrace{\left(i\partial_t - \frac{\nabla^2}{2} + \mu\right)}_{H_0} \psi(x) \; - \; g \underbrace{\int \psi^\dagger_\uparrow(x)\, \psi^\dagger_\downarrow(x)\, \psi_\downarrow(x)\, \psi_\uparrow(x)}_{\text{4-fermion operator}}$$

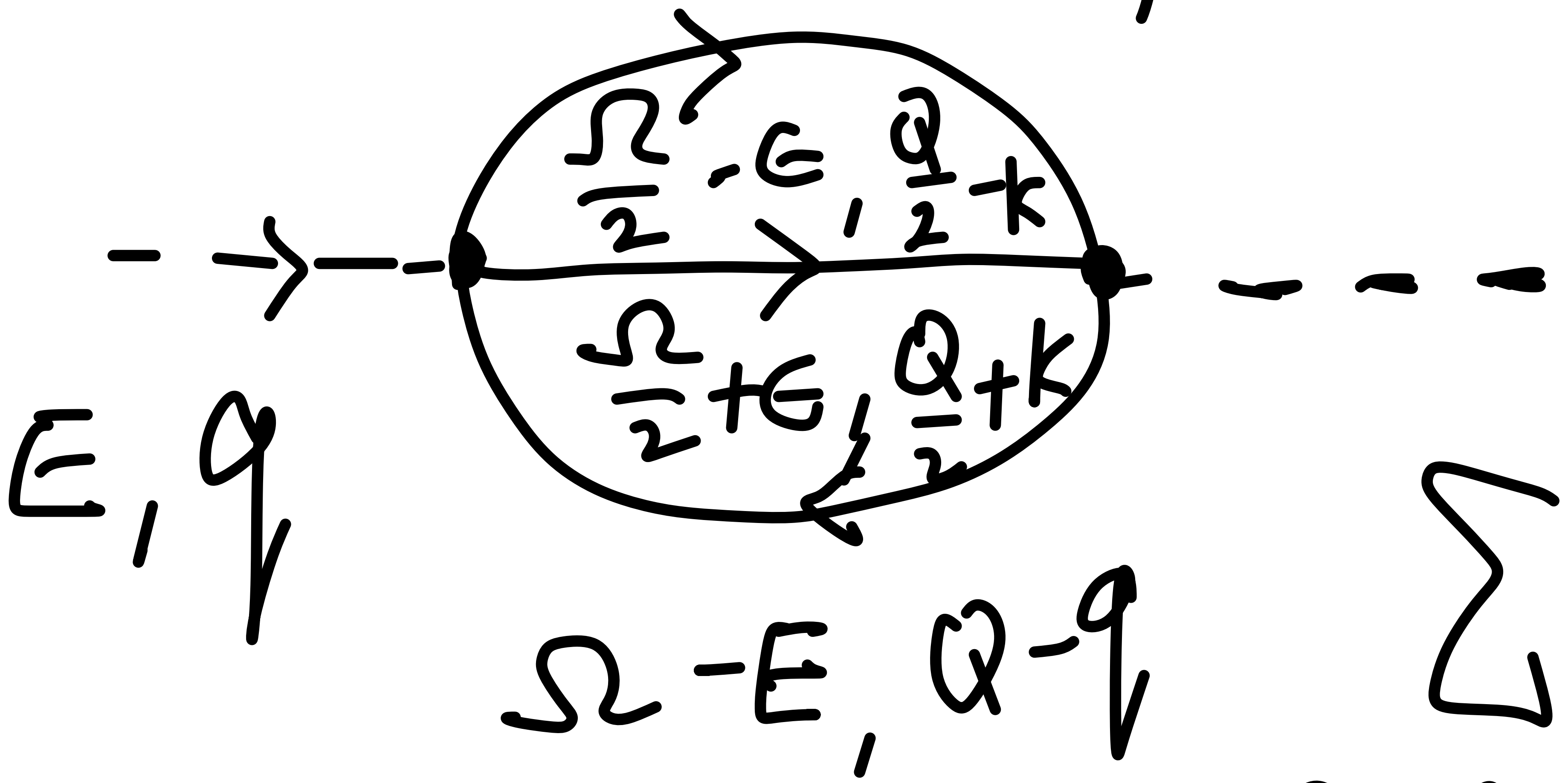
$(g > 0)$



$\mu = 0$

$E - (\varepsilon_2 + \varepsilon_1)$, $\vec{Q} - (\vec{k}_1 + \vec{k}_2)$

$$G(\varepsilon, k) \simeq \frac{1}{\varepsilon - \varepsilon_k + i\delta}, \quad \delta > 0$$



$$\sum_{\Omega, \epsilon, Q, k} G\left(\frac{\Omega}{2} - \epsilon, \frac{Q}{2} - k\right) G\left(\frac{\Omega}{2} + \epsilon, \frac{Q}{2} + k\right) G(\Omega - E, Q - q)$$

$$= \sum_{Q, k} \frac{1}{\Omega - \frac{Q^2}{4} - k^2 + i\delta} \; \frac{1}{\Omega - E - \frac{(Q-q)^2}{2} + i\delta}$$

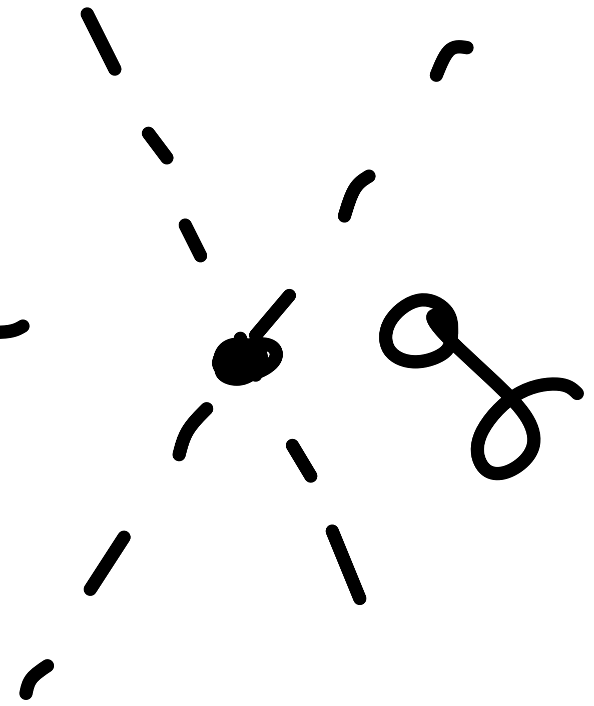
$= 0$

Conclusion:

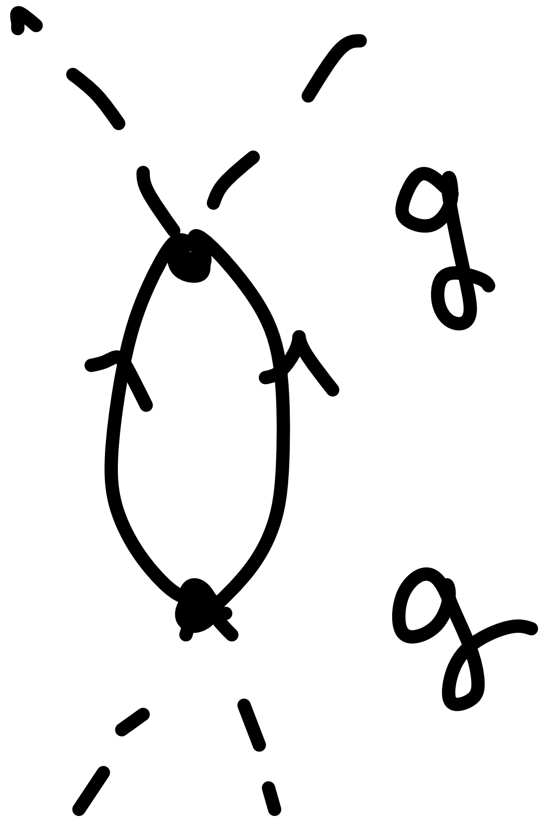
$$\frac{1}{Z}\frac{dZ}{dt} = \gamma(\tilde{g}) = 0 \quad \text{when } \mu = 0$$

$$\left[\beta(g)\frac{\partial}{\partial g}+\frac{\partial}{\partial t}\right]G^{(4)}=0\;,$$

$G^{(4)} =$



tree level

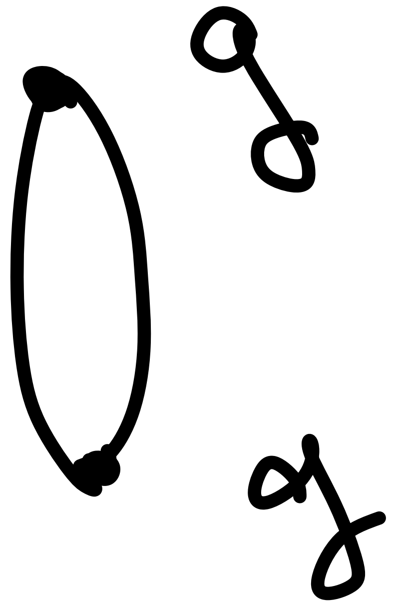


One-loop

$$\delta g(\Lambda)=\frac{g^2}{-i}\int\frac{d\omega}{2\pi}\int\frac{d^d\vec{k}}{(2\pi)^d}\,G(\omega,\vec{k})\,G(-\omega,-\vec{k})$$

$$=\int_0^\Lambda\frac{d^d\vec{k}}{(2\pi)^d}\left[\frac{1}{-2\varepsilon_k}\right]g^2$$

$\beta(g) = -\frac{\partial}{\partial t}$



$$= \frac{\Omega_d}{(2\pi)^d}\,g^2\,\Lambda^{d-2}$$

$$\tilde{g} = g\Lambda^{d-2}, \qquad \frac{dg}{dt} = \beta(g) = C_d\, g^2 \Lambda^{d-2}$$

$$\beta(\tilde{g}) \doteq \frac{d\tilde{g}}{dt} = (d-2)\tilde{g} + \Lambda^{d-2}\beta(g)$$

$$= (d-2)\tilde{g} + \tilde{g}^2 C_d$$

$d=3$, FPs

$$\beta(\tilde{g}=0)=0, \qquad \frac{d\tilde{g}}{dt} \doteq (d-2)\tilde{g} = \tilde{g},$$

$$\longrightarrow \quad \tilde{g}(\Lambda) \sim \left(\frac{\Lambda}{\Lambda_{UV}}\right)^1 \tilde{g}_{UV}, \qquad \tilde{g}: \text{irrelavent, in IR}$$

$d=3, \quad \beta(\tilde{g}=\tilde{g}^*)=0, \quad \tilde{g}^* = (2-d)\, C_d^{-1}.$

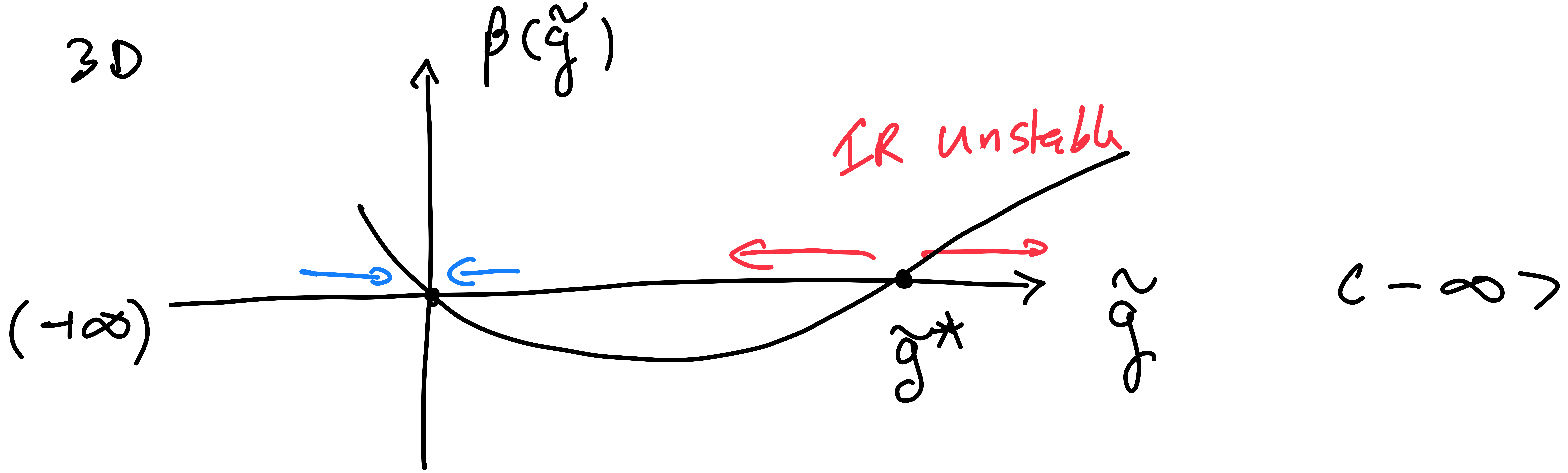
$$\frac{d\delta\tilde{g}}{dt} = (2-d)\,\delta\tilde{g}, \qquad \delta\tilde{g}(\Lambda) \simeq \frac{\Lambda_{uv}^{2-d}}{\Lambda^{2-d}}\,\delta\tilde{g}_{uv}$$

$\delta\tilde{g}(\Lambda)$ Relavent in the IR limit.

Two FPs : $\left\{\begin{array}{l} \tilde{g}=0, \quad \text{IR stable} \\ \tilde{g} = (2-d)\, C_d^{-1}, \quad \text{IR unstable.} \end{array}\right.$

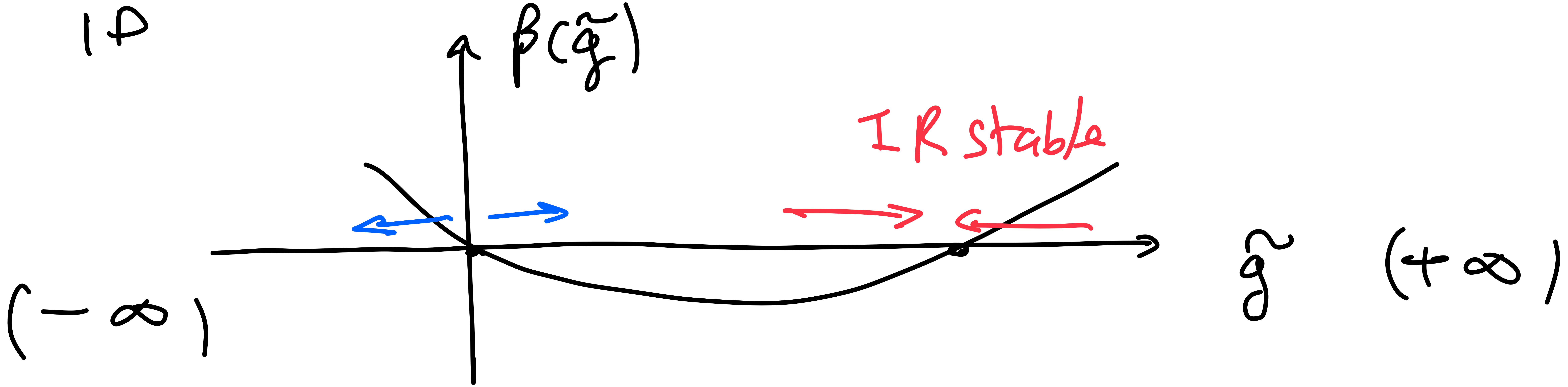
$(d=3)$

3D



$\beta(\tilde{g})$
IR unstable
$(+\infty)$
$\tilde{g}^*$
$\tilde{g}$
$(-\infty)$

1D



$\beta(\tilde{g})$
IR stable
$(-\infty)$
$\tilde{g}$
$(+\infty)$