Phys525:
Quantum Condensed Matter Physics:
Emergent phenomena and emergent symmetries

Episode 13:
Scale transformation and QFTs with **scale** symmetries as QCPs I:
Basic Phenomenology (informal discussions)
and identifications of scale invariant fixed Points (**SIFP**)

(QFT CS-RGE based slightly formal discussions next Thursday)

Fluctuations near $\langle \phi \rangle$, $\quad \phi \to \langle \phi \rangle + \phi$

$$\mathcal{L}(\{\phi(\vec{r},\tau)\}) = |\partial_\tau \phi|^2 + |\nabla \phi|^2 + m^2 |\phi|^2 + \dots, \quad m^2 = 4|\alpha|$$

$$\langle \phi(\vec{r},\tau)\, \phi(0,0) \rangle \simeq \int \frac{d^d k}{(2\pi)^d} \int \frac{d\omega}{2\pi}\, G(\omega, \vec{p})\, e^{i \vec{p}\cdot\vec{r} - i\omega\tau}$$

$$C(\vec{r},\tau) \sim \frac{1}{|\vec{R}|^{d-1}}\, e^{-\frac{|\vec{R}|}{\xi_c}}, \quad \underbrace{\vec{R} = (\vec{r},\tau)}_{(d+1)\ \text{dimension}}$$

$$\xi_c \sim \frac{1}{m} \sim \frac{1}{|J - J_c|^{\boxed{1/2}}}$$

$$(\, |\alpha| \propto J - J_c > 0 \,)$$

$$G.C.^{(d+1)} = \frac{\left\langle \frac{1}{V_c}\int_{V_c} \tilde{\Phi}^*(\vec{R}')\, d^{(d+1)}\vec{R} \cdot \frac{1}{V_c}\int_{V_c} \tilde{\Phi}(\vec{R})\, d^{(d+1)}\vec{R} \right\rangle}{\Delta_0^2}$$

$$= \frac{1}{\Delta_0^2 V_c^2}\int_{V_c} \left\langle \tilde{\Phi}^*(\vec{R}')\, \Phi(\vec{R}) \right\rangle d^{(d+1)}\vec{R}'\, d^{(d+1)}\vec{R}$$

$$\searrow \quad G(\vec{R}' - \vec{R}) \sim \frac{1}{|\vec{R}' - \vec{R}|^{d-1}}$$

$$\sim \frac{1}{\Delta_0^2}\, \frac{1}{\xi_c^{d-1}} \sim \xi_c^{3-d}$$

$$\left(V_c \sim \xi_c^{d+1} , \quad \Delta_0^2 \sim |\alpha| \sim \xi_c^{-2} \right)$$

$$\Delta_0 \sim \frac{1}{\xi_c}$$

Important Statements on fluctuations ← Quantum

$G^{Q} = G_{\circ} C_{\circ}^{(d+1)} \sim \xi_c^{\,3-d}$ ($\xi_c \to \infty$ at critical points)

1) $d > 3$, G^{Q} approaches zero as $\xi_c \to \infty$.
fluctuations appear to be irrelevant.

2) $d < 3$, G^{Q} diverges as $\xi_c \to \infty$
fluctuations appear to be irrelevant
and MF can be qualitatively problematic.

$d = 3$ Upper critical dimension for Q.C.P. !!

We are here.

Caracas
VENEZUELA
Georgetown
GUYANA
Atlantic Ocean
Bogotá
COLOMBIA
Quito
ECUADOR
Vaupés
Caquetá
Apaporis
Negro
Branco
Japurá
Putumayo
Napo
Tigre
Pastaza
Iquitos
Marañón
Trombetas
Paru
Jari
Macapá
Para
Belém
Sanatarém
Manaus
Javary
Juruá
Purus
Madeira
Tapajós
Iriri
Imperatriz
Huallaga
Ucayali
Juruena
Teles Pires
Xingu
Pucallpa
Rio Branco
Porto Velho
Machado
Arinos
Araguaia
Tocantins
Acre
Madre de Dios
Tambo
Ene
Lima
Mantaro
Guaporé
Apurímac
Cusco
Beni
Mamoré
PERU
La Paz
Grande
Brasília
Cochabamba
Santa Cruz
BRAZIL
Pacific Ocean
BOLIVIA
Rio de Janeiro
São Paulo

- Two corner stones of EFT approach to QCPs

- **Symmetry groups, space-time symmetry (as well as PHC) and the corresponding QFT.** G= Z2, U(1), SU(N), N=2, 3, 4… [with caveats: QFTs can further depend on symmetry group representations or *with quantum anomalies as research focuses.*]

- **Scale Symmetric QFTs or fixed points (SIFPs).**

1. Wilson, K. G. (1971). "Renormalization Group and Critical Phenomena. I. Renormalization Group and the Kadanoff Scaling Picture". Physical Review B. **4** (9): 3174–3183. Bibcode:1971PhRvB...4.3174W. doi:10.1103/PhysRevB.4.3174.
2. ^ Wilson, K. (1971). "Renormalization Group and Critical Phenomena. II. Phase-Space Cell Analysis of Critical Behavior". Physical Review B. **4** (9), 3184.
3. Wilson, K. (1983). "The renormalization group and critical phenomena". Reviews of Modern Physics. **55** (3): 583–600.

Scale transformation and Scale symmetry: Phenomenology and informal stuff (without RGE)

- Scale transformation in QFT (in real space)

- Definition of scale invariance in terms of (d+1) dimension action/Lagrangian density

- How to identify SIFP.

Scale transformation and Scale symmetry: Phenomenology without RGE

- Scale transformation in QFT (real space phenomenology)

- Definition of scale invariance in terms of (d+1) dimension action/Lagrangian density

What does Scale symmetry imply at QCPs ?

Scale transformation :

$$\vec{r}' = \vec{r}\lambda , \quad \tau' = \tau \lambda^{\underbrace{z}_{\text{dynamical critical Exponent}}} , \quad \psi'(\vec{r}', \tau') = \lambda^{\underbrace{-\eta}_{\text{Scale dimension of field "}\psi\text{"}}} \psi(\vec{r}, \tau)$$

(for MI-SF QCP at $J = J_c$, $z = 1$, $\eta = \frac{d-1}{2}$ MF fixed pt)

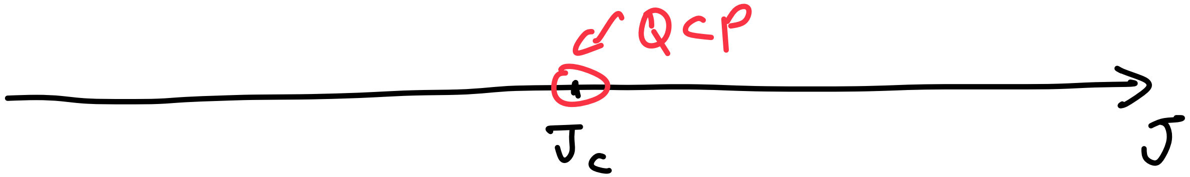
Scale invariance at QCPs

$$S(\{\psi(r,\tau)\}) \longrightarrow S'(\{\psi'(\vec{r}', \tau')\}) = S(\{\psi'(\vec{r}', \tau')\})$$

or

$$\mathcal{H}(\{b^{+}_{\vec{r}}, b_{\vec{r}}\}) \longrightarrow \mathcal{H}'(\{b^{+\prime}_{\vec{r}'}, b'_{\vec{r}'}\}) = \lambda^{-z} \mathcal{H}(\{b^{+}_{\vec{r}'}, b_{\vec{r}'}\})$$

$z=1$ Example I.



$c=1,$

$$\mathcal{L} = |\partial_\tau \varphi|^2 + |\nabla \varphi|^2 + m^2|\varphi|^2 + \underbrace{\cdots}_{\text{irrelavent } d>3}$$

$$\mathcal{L}_{QCP} = \mathcal{L}(m=0)$$

$$\tau \to \tau' = \tau\lambda \quad x \to x' = x\lambda, \quad \varphi \to \varphi'(x') = \lambda^{-\frac{d-1}{2}}\varphi(x,\tau)$$

$$\mathcal{L} = \{|\partial_{\tau'}\varphi'|^2 + |\nabla_{\vec{r}'}\varphi'|^2\}\lambda^{+\frac{d-1}{2}\times 2+2} + m^2|\varphi'|^2\lambda^{+(\frac{d-1}{2})\times 2}$$

$$S = \int d^{(d)}x\, d\tau\, \mathcal{L} \longrightarrow S' = \int d^{(d)}x' d\tau'$$

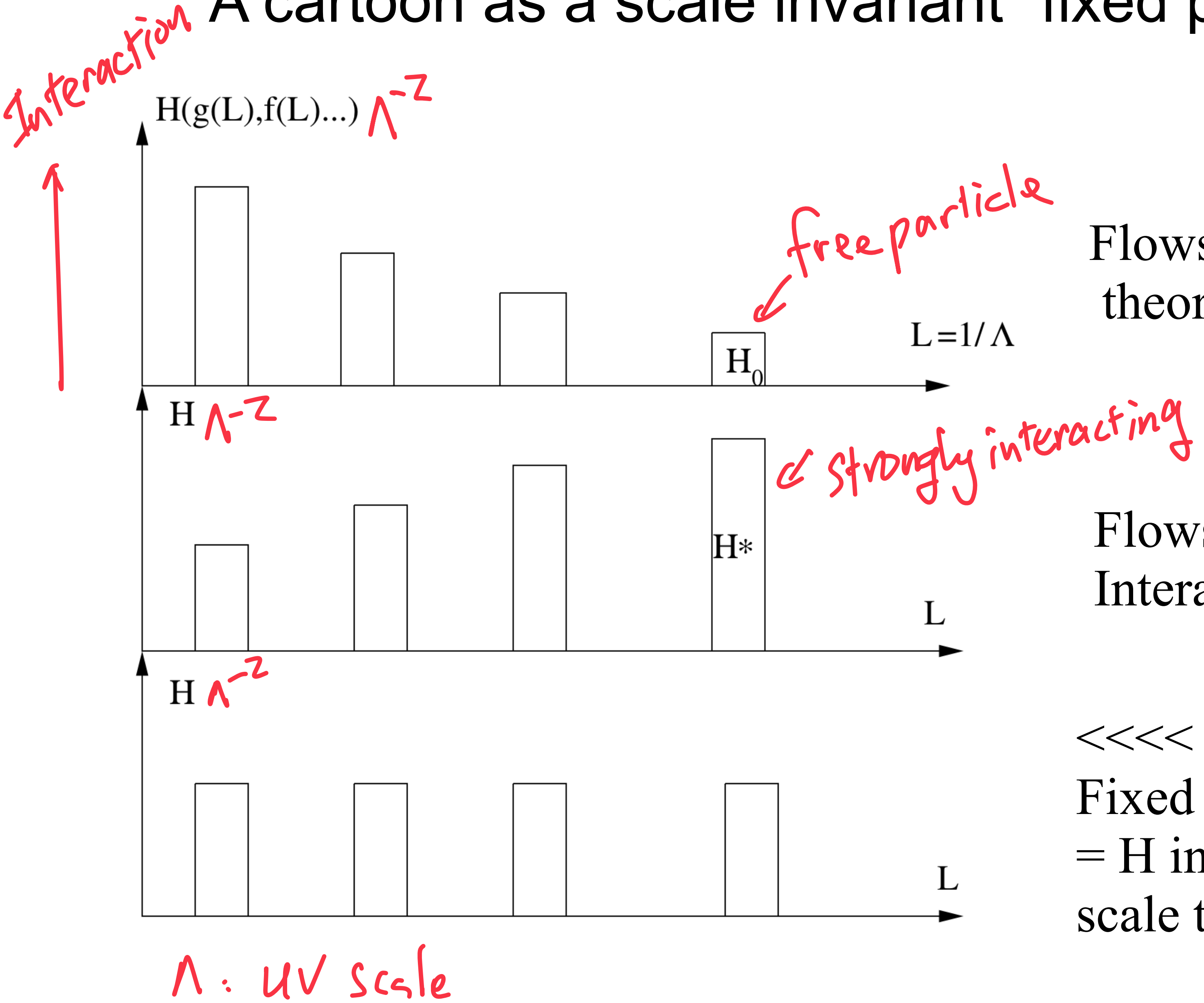
$$\{[|\partial_{\tau'}\varphi'|^2 + |\nabla_{\vec{r}'}\varphi'|^2] + \lambda^{-2} m^2|\varphi'|^2\}$$

breaks scale symmetry

So that $S \longrightarrow S' = S(\{\varphi'(\vec{r}', \tau')\})$ if $m=0$ or QCP!

Scale invariant.

Wilson’s view of critical points (70s): A cartoon as a scale invariant “fixed point”



Flows into a free theory

Flows into a strongly Interacting theory

<<<<< Critical point/ Fixed Point = H invariant under scale transformation

Scale transformation: General

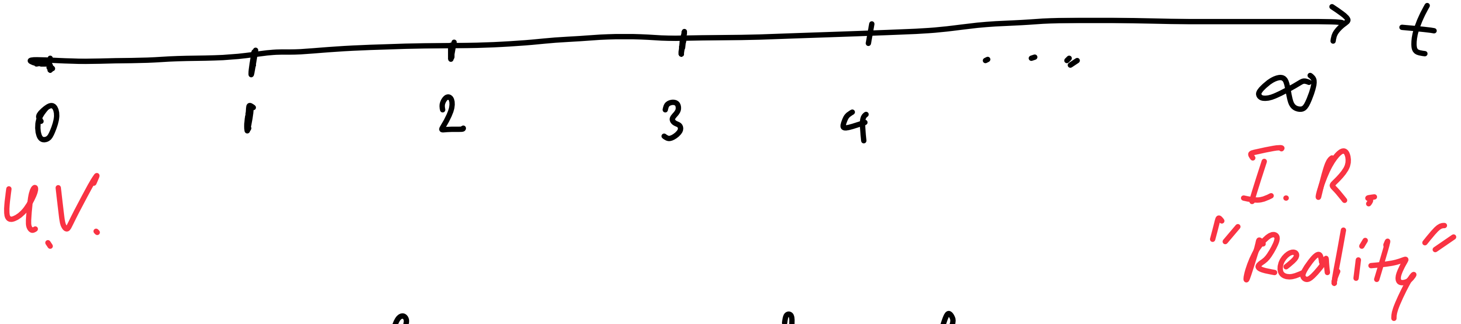
$\ln\lambda = t$ or $\lambda = e^{t}$

best defined in terms of t
Not λ

$$\tau \to \tau' = \tau e^{-tz}$$

$$\vec{x} \to \vec{x}' = \vec{x} e^{-t}$$

$$\varphi(\vec{x},t) \to \varphi'(\vec{x}',\tau') = e^{+\eta t}\varphi(\vec{x},t), \quad \left[\varphi(\vec{x},\tau) = e^{-\eta t}\varphi_t(\vec{x}_t,\tau_t)\right]$$



$$\mathcal{L}_0(\{\varphi(\vec{x},t)\}) \to \mathcal{L}_{t=1}(\{\varphi_1(\vec{x}_1,t_1)\} \to \mathcal{L}_{t=2}(\{\varphi_2(x_2,t_2)\}) \to \dots$$

bare field
$UV = \Lambda_{UV}$

Renormalized field
with $UV = \Lambda_{UV} e^{-1}$

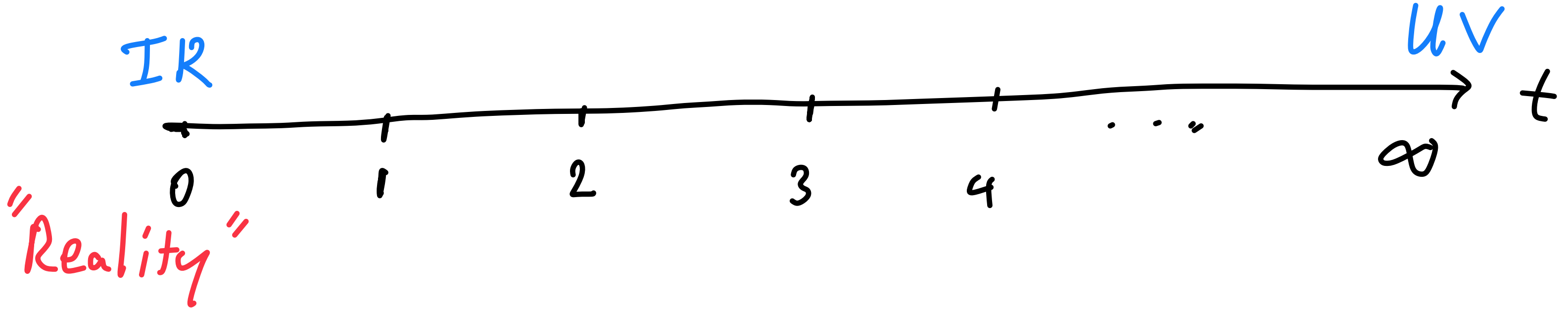
Renormalized field
with $UV = \Lambda_{UV} e^{-2}$

Scale transformation: General II: "particle physics" towards unification

$$\tau \to \tau' = \tau e^{tz}$$

$$\vec{x} \to \vec{x}' = \vec{x} e^{t}$$

$$\varphi(\vec{x}, t) \to \varphi'(\vec{x}', \tau') = e^{-\eta t} \varphi(\vec{x}, t), \qquad \varphi(\vec{x}, \tau) = e^{+\eta t} \varphi_t(\vec{x}_t, \tau_t)$$



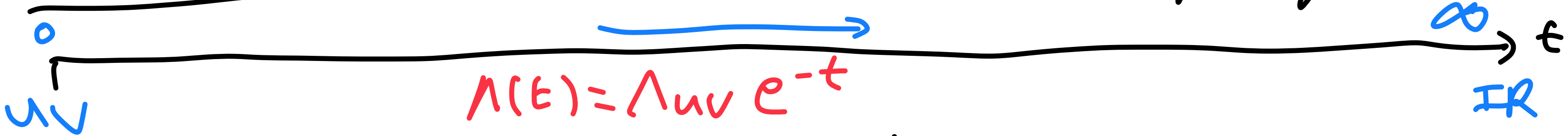
$$\mathcal{L}_{IR}(\{\varphi(\vec{x}, t)\}) \to \mathcal{L}_{t=1}(\{\varphi_1(\vec{x}_1, t_1)\} \to \mathcal{L}_{t=2}(\{\varphi_2(x_2, t_2)\}) \to \ldots$$

$\swarrow$ Renormalized field

$\swarrow$ Renormalized field with $UV = \Lambda_{IR} e^{+1}$

$\swarrow$ Renormalized field with $UV = \Lambda_{IR} e^{+2}$

Scale transformation and Renormalization Group equations



$\Lambda(t) = \Lambda_{uv}\, e^{-t}$

$$\mathcal{L}_t(\{\varphi(\vec{x},\tau)\}; \overset{\nearrow}{m}, \overset{\searrow}{g}, \dots) = \Lambda(t)^{d+1}\, \tilde{\mathcal{L}}_t(\{\varphi(x,\tau)\}; \tilde{m}, \tilde{g}, \dots)$$

Mass

interactions

$\propto S$

$\downarrow$

define at $\Lambda(t) = \Lambda_{uv}\, e^{-t}$

or a running scale

$$\underbrace{\tilde{m}(t) = m(t)\, \Lambda(t)^{-z}, \quad \tilde{g}(t) = g(t)\, \Lambda(t)^{d-3}}\quad \dots$$

transforms in unique way under Scale Transformation

follows a set of PDE the called "R.G.E."

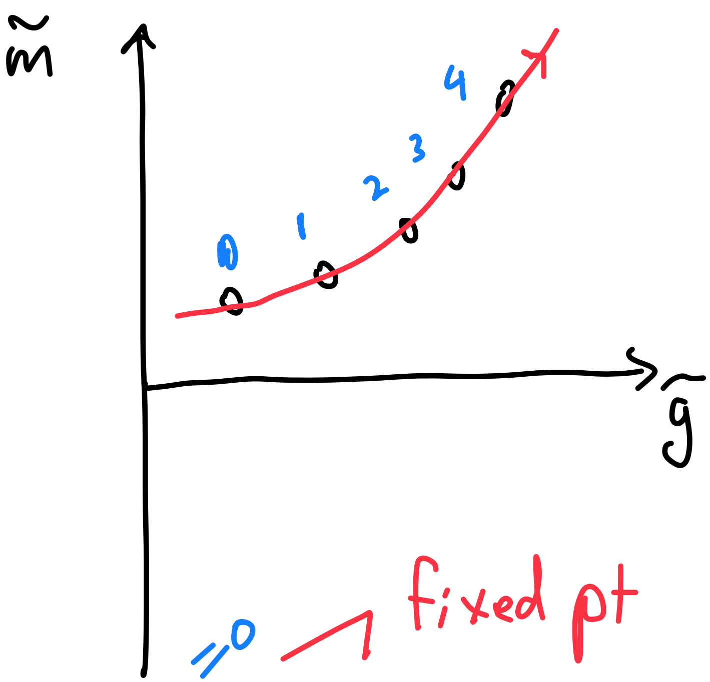
Scale transformation and Scale symmetry: Phenomenology without RGE

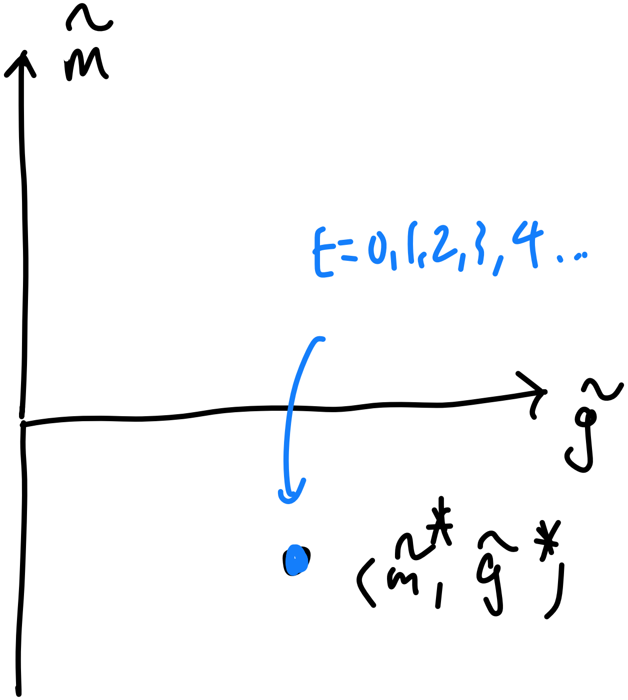
- Scale transformation in real space
- Definition of scale invariance in terms of (d+1) dimension action/Lagrangian density
- How to identify SIFP.

RGE

$$\frac{d\tilde{m}}{dt} = \beta_m(\tilde{m}, \tilde{g}, \ldots), \qquad \frac{d\tilde{g}}{dt} = \beta_g(\tilde{m}, \tilde{g}, \ldots)$$

A) Generically, $\beta_m \neq \beta_g \neq 0$. and $\tilde{m}(t), \tilde{g}(t)$ are running.



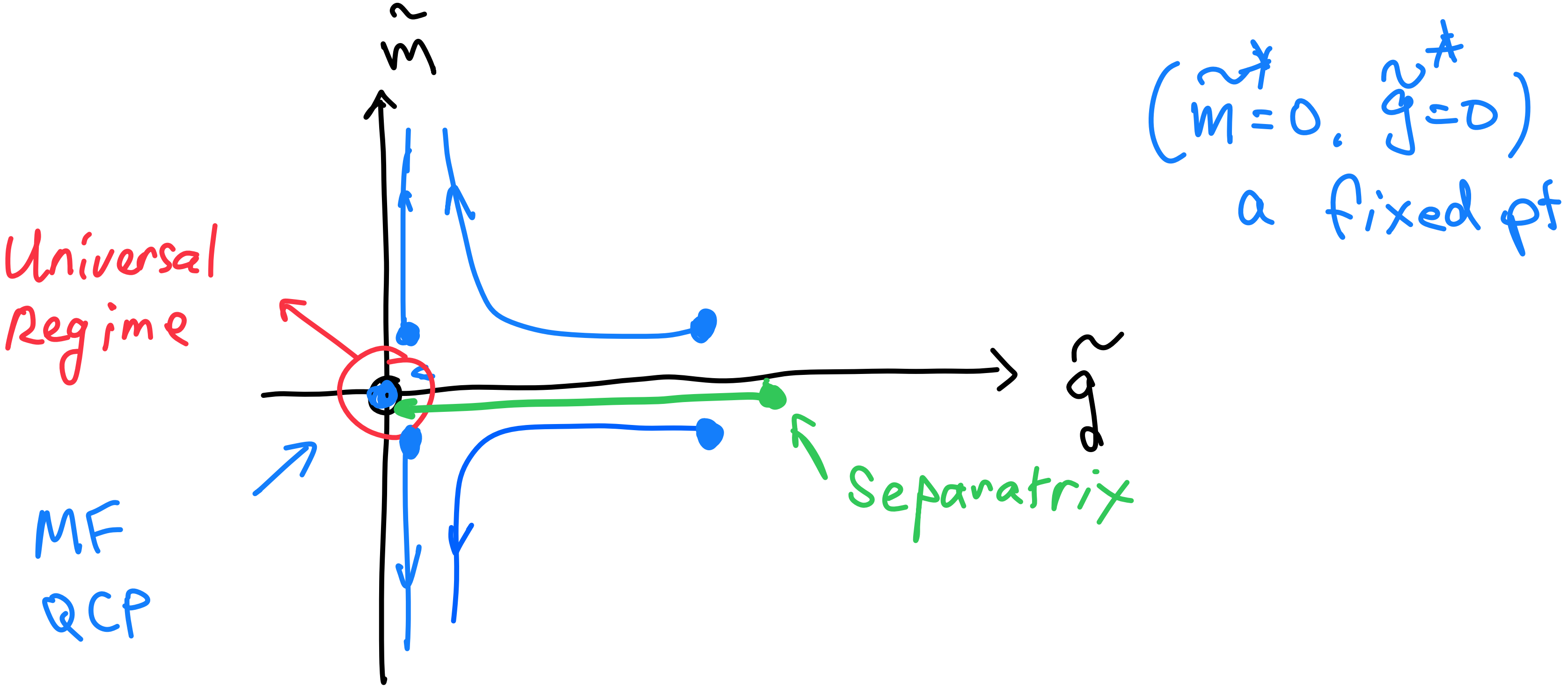


fixed pt

B)

if $\overset{=0}{\beta_m(\tilde{m}^*, \tilde{g}^* \ldots)} = \overset{=0}{\beta_g(\tilde{m}^*, \tilde{g}^*.)}$, $\tilde{m}(t), \tilde{g}(t)$ Not Running

$d>3$, QCP for MI-SF transition at $J=J_c$

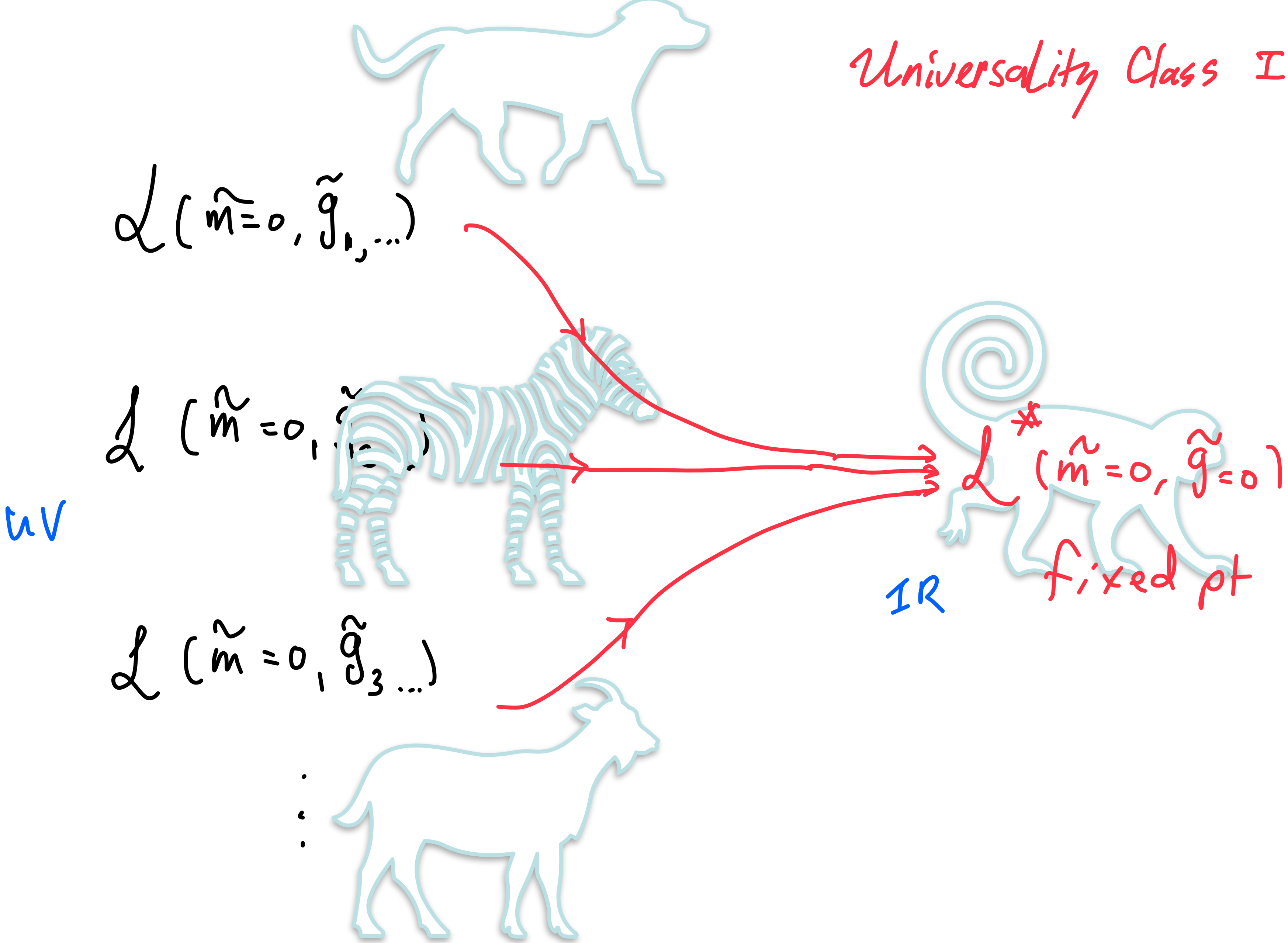


$$\mathcal{L} = |\partial_t\varphi|^2 + |\tilde{\nabla}\varphi|^2 + m^2|\varphi|^2 + g|\varphi|^4 + \cdots$$

fixed pt

$$\mathcal{L}\,(m=0, g\neq0, \ldots) \xrightarrow{\text{Flow into}} \mathcal{L}^*(m=0, g=0, 0, 0, \ldots)\,\mathcal{L}^*$$

Universality Class I



$\mathcal{L}(\tilde{m}=0, \tilde{g}_1, \dots)$

$\mathcal{L}(\tilde{m}=0, \tilde{g}_2 \dots)$

$\mathcal{L}(\tilde{m}=0, \tilde{g}_3 \dots)$

$\vdots$

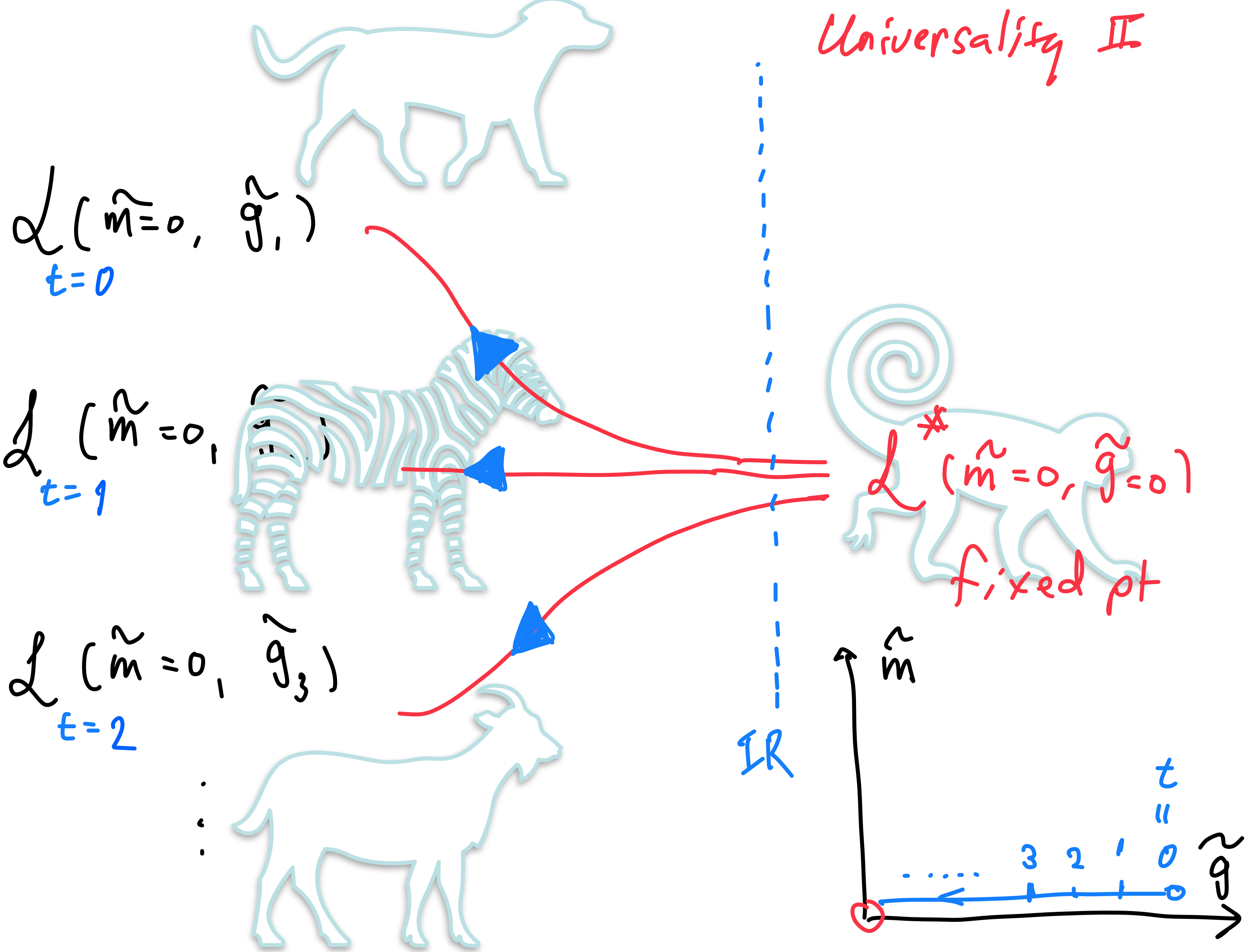
UV

IR

$\mathcal{L}^*(\tilde{m}=0, \tilde{g}=0)$

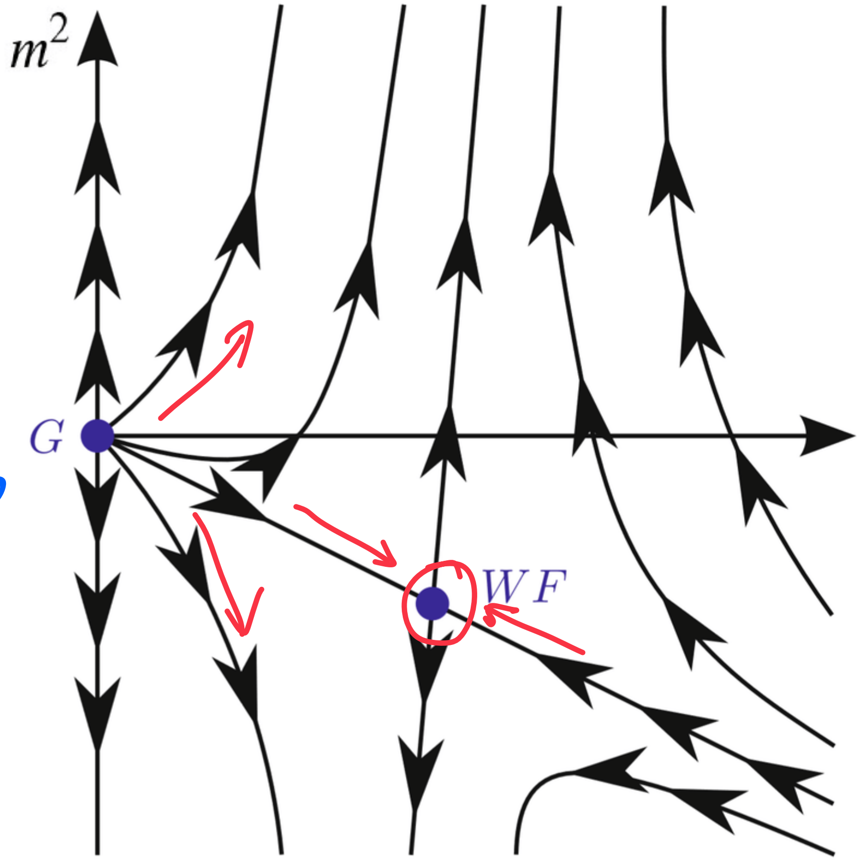
fixed pt

Universality II



RG flows (i.e. scale transformation) of a scalar model : Scale Symmetric Wilson-Fisher F.P. (d+1<4)

$d < 3$



mass

$\lambda = g$ interaction

$g\phi^4$

MF fixed pt

IR unstable

WF fixed pt $(d<3)$

I.R. Stable

Scale invariance and Phase Transitions

1) Effective QFT or EFT constructed out of symmetry and other general considerations.

1) Critical point is identified as a scale invariant QFT (or CFT if z=1,2) or a fixed point Hamiltonian understand scale transformation.

3) Microscopic information (which appears in EFT) wiped out in the course of renormalization leading to universalities.