Phys525:
Quantum Condensed Matter Physics: Quantum Criticality
Basics, Dynamics and Topological states

Episode 12: Correlation function in “Z” Vs **Time Ordering** in quantum Ground states:

Slightly more formal stuff on Space-time temporal correlations

imaginary time evolution (QFT: Vac-Vac Amplitude)

- Extract ground state properties (today)
- Extract space-time symmetries suitable for EFTs (today)
- Evaluate the dynamic correlations

From classical correlation functions to causal Green's functions (retarded and advanced)

$$C(r,\tau;0{,}0) = < \psi(0{,}0)\psi^*(r,\tau) >_{Z_{qft}(D=d+1)}$$

$$\rightarrow G(\vec{r},\tau;0{,}0) = < g\,.\,s\,.\,| - i\mathscr{T}\, b(\vec{r},\tau) b^{\dagger}(0{,}0)\,|\, g\,.\,s\,. >$$

$$\rightarrow G^{R}(\vec{r},t;0{,}0) = < g\,.\,s\,.\,| - i[b(\vec{r},t), b^{\dagger}(0{,}0)]\,|\, g\,.\,s\,. > \theta(t)$$

$$\rightarrow G^{A}(\vec{r},t;0{,}0) = < g\,.\,s\,.\,| - i[b(\vec{r},t), b^{\dagger}(0{,}0)]\,|\, g\,.\,s\,. > \theta(-t)$$

$$C(\tau) = iG(\tau = it); \qquad\qquad C(\Omega) = -\,G(i\Omega);$$

$$G(i\Omega \to \Omega + i\delta) = G^R(\Omega)$$

$$\to G^R(\Omega) \to G^R(\vec{r}, t)$$

$$Z = \langle 0 | e^{-HT} | 0 \rangle$$

$$= \int D\phi \; e^{-\underbrace{\int d^{(d)}\vec{r}\, d\tau\, \mathcal{L}(\{\phi(\vec{r},\tau)\})}_{S(\{\phi(\vec{r},\tau)\})}}$$

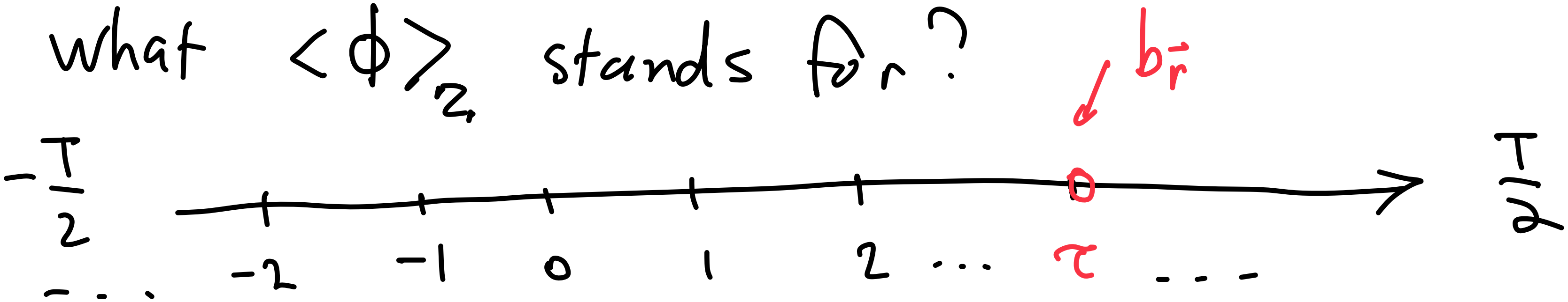
$$E_{g.s.} \overset{T\to\infty}{=} -\frac{1}{T} \ln Z$$

$$\langle \phi \rangle_Z = \frac{\int D\phi \; \phi(\vec{r},\tau) e^{-S(\{\phi(\vec{r},\tau)\})}}{\int D\phi \; e^{-S(\{\phi(\vec{r},\tau)\})}}$$

$$\langle \phi^*(\vec{r},\tau) \phi(0,0) \rangle_Z = \frac{\int D\phi \; \phi^*(\vec{r},\tau)\phi(0,0) e^{-S}}{\int D\phi \; e^{-S}}$$

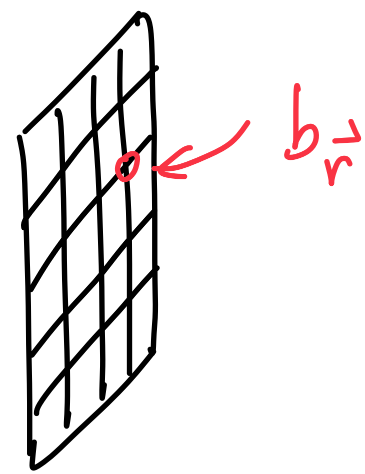
$$\langle \phi \rangle_\tau = \frac{\int D\phi \; \phi(\vec{r},\tau)\, e^{-S(\{\phi(\vec{r},\tau)\})}}{\int D\phi \; e^{-S(\{\phi(\vec{r},\tau)\})}}$$

what $\langle \phi \rangle_\tau$ stands for?



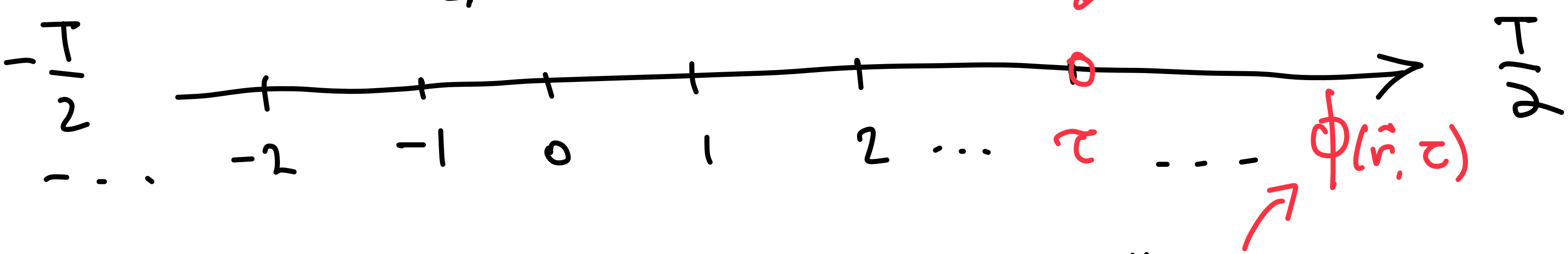
$$b_{\vec{r}} |\phi_{\vec{r}}\rangle = \phi(\vec{r}) |\phi_{\vec{r}}\rangle$$

"Coarse grained state"



$$\langle \phi \rangle_Z = \frac{\int \Delta\phi \ \phi(\vec r,\tau)\, e^{-S(\{\phi(\vec r,\tau)\})}}{\int \Delta\phi \ e^{-S(\{\phi(\vec r,\tau)\})}}$$

what $\langle \phi \rangle_{Z}$ stands for?



$$Z_{b_{\vec r}} = \cdots \sum_{\{\phi(\vec r,\tau+1)\}} \ \sum_{\{\phi(\vec r,\tau)\}} \cdots \langle \{\phi(\tau+1)\}|\, e^{-H\Delta\tau}\, b_{\vec r}\, |\{\phi(\tau)\}\rangle \cdots$$

$$= \langle 0|\, \underbrace{e^{-H(\frac{T}{2}-\tau)}}_{\text{After}}\, b_{\vec r}\, \underbrace{e^{-H(\tau+\frac{T}{2})}}_{\text{before}}\, |0\rangle$$

what $\langle \phi \rangle_2$ stands for?

$b_{\vec{r}}$

Imaginary time

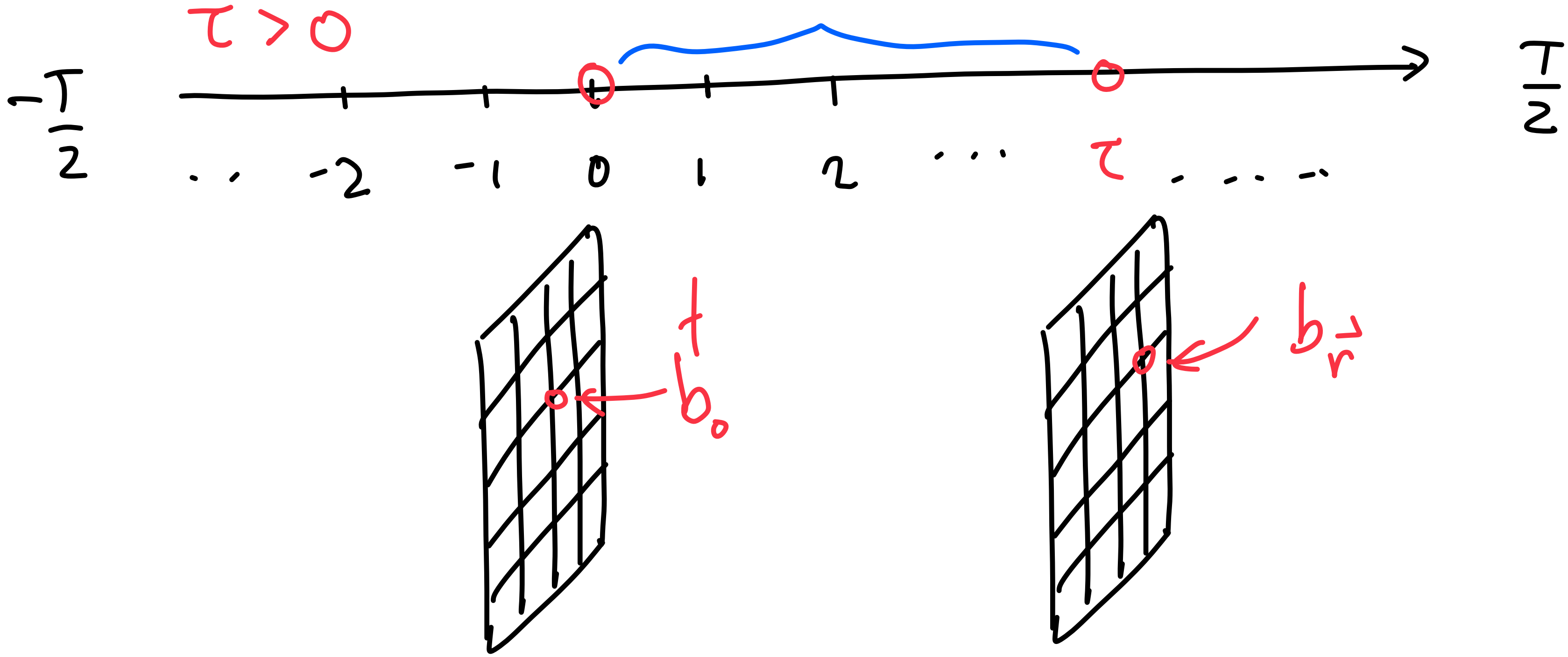
$\cdots \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad \cdots \quad \tau \quad \cdots$

$$\langle \phi \rangle_2 = \frac{Z(b_{\vec{r}}, \tau)}{Z} = \frac{\langle 0 | e^{-(\frac{T}{2}-\tau)H} \, b_{\vec{r}} \, e^{-(\tau+\frac{T}{2})H} | 0 \rangle}{\langle 0 | e^{-TH} | 0 \rangle}$$

$$\overset{T\to\infty}{\simeq} \frac{|\langle 0 | g.s. \rangle|^2 e^{-E_{g.s.}T}}{|\langle 0 | g.s. \rangle|^2 e^{-E_{g.s.}T}} \cdot \langle g.s. | b_{\vec{r}} | g.s. \rangle$$

g.s. Quantum Average!

Averge over "Z" vs Quantum State averge



$$\langle \phi^*(\vec r,\tau)\,\phi(0,0)\rangle_Z \overset{T\to\infty}{=} \langle g.s.|\,\underbrace{e^{H\tau}\, b_{\vec r}\, e^{-H\tau}}_{b(\vec r,\tau)}\, b_0^\dagger(0)\,|g.s.\rangle$$

(Heisenberg Rep.)

$\tau > 0$

$$\langle \phi(\vec{r},\tau)\, \phi^{*}(0,0) \rangle_{Z} \xrightarrow{T\to\infty} \langle g.s.|\, \underbrace{e^{H\tau}\, b_{\vec{r}}\, e^{-H\tau}}_{b^{\dagger}(\vec{r},\tau)}\, b_{0}^{\dagger(0)} |g.s.\rangle$$

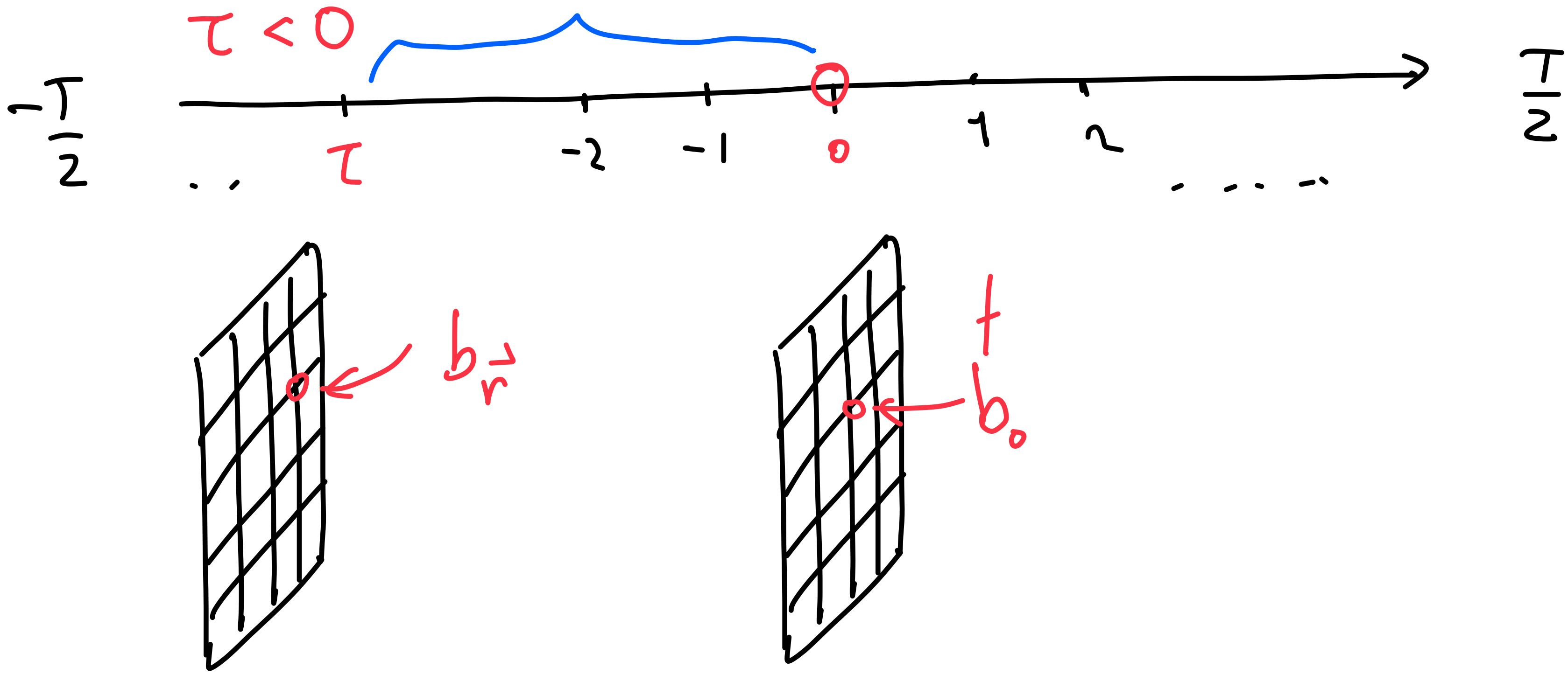
(Heisenberg Rep.)

$$\langle g.s.|\, b(\vec{r},\tau)\, b_{0}^{\dagger} |g.s.\rangle$$

All excitations

$$= \sum_{n} \langle g.s.|\, b(\vec{r},\tau) |n\rangle \langle n|\, b_{0}^{\dagger} |g.s.\rangle$$

Dynamic Correlation function $\longleftrightarrow$ excitations



$$\langle \phi(\vec r,\tau)\,\phi^*(0,0)\rangle_Z \overset{T\to\infty}{=} \langle g.s.|\, b_0^\dagger\, e^{-H(0-\tau)}\, \underbrace{b_{\vec r}\, e^{-H\tau}}_{b(\vec r,\tau)}\, |g.s.\rangle$$

$\tau < 0$

(Heisenberg Rep.)

TIME ORDERING

$\tau > 0$

$$\langle \phi^*(\vec{r},\tau)\,\phi(0,0)\rangle_Z \overset{T\to\infty}{=} \langle g.s.|\underbrace{e^{H\tau}\, b_{\vec{r}}\, e^{-H\tau}}_{b(\vec{r},\tau)}\, b_0^{(0)\dagger}|g.s.\rangle$$

$\tau < 0$

$$\langle \phi^*(\vec{r},\tau)\,\phi(0,0)\rangle_Z \overset{T\to\infty}{=} \langle g.s.|\, b_0^\dagger\, \underbrace{e^{H\tau}\, b_{\vec{r}}\, e^{-H\tau}}_{b(\vec{r},\tau)}|g.s.\rangle$$

Pull together,

$$\langle \phi^*(\vec{r},\tau)\,\phi(0,0)\rangle_Z = \langle g.S.|\,\mathcal{T}\, b_{\vec{r}}(\tau)\, b_0^{(0)\dagger}|g.s.\rangle$$

$$\mathcal{T}\, b_{\vec{r}}(\tau)\, b_0^\dagger(0) = \Theta(\tau)\, b_{\vec{r}}(\tau)\, b_0^\dagger + \Theta(-\tau)\, b_0^\dagger\, b_{\vec{r}}(\tau)$$

Fluctuations near $\langle \phi \rangle$, $\quad \phi \to \langle \phi \rangle + \phi$

$$\mathcal{L}(\{\phi(\vec r, \tau)\}) = |\partial_\tau \phi|^2 + |\nabla \phi|^2 + m^2 |\phi|^2 + \dots, \quad m^2 = 4|\alpha|$$

$$Z = \int D\phi_{\vec r, \tau}\, e^{-\int d^{(d)}\vec x\, d\tau\, \mathcal{L}(\{\phi(\vec r, \tau)\})}$$

$$\simeq \int D\phi_{\omega, q}\, e^{-\sum_\omega \sum_q \phi^*(\omega, \vec q)\, G^{-1}(\omega, \vec q)\, \phi(\omega, \vec q)} = \prod_{\omega, q} Z_{\omega, \vec q}$$

MF

$$G^{-1}(\omega, q) = \omega^2 + \vec q^{\,2} + m^2$$

$$\langle \phi^*(\omega, \vec q)\, \phi(\omega, \vec q) \rangle \simeq \mathrm{X}\; G(\omega, \vec q)$$

$Z_{\omega, \vec q}$ $\quad$ HW Set II

Time ordered in Non-Relativistic Theory (free particles)

$|0\rangle = |vac\rangle$

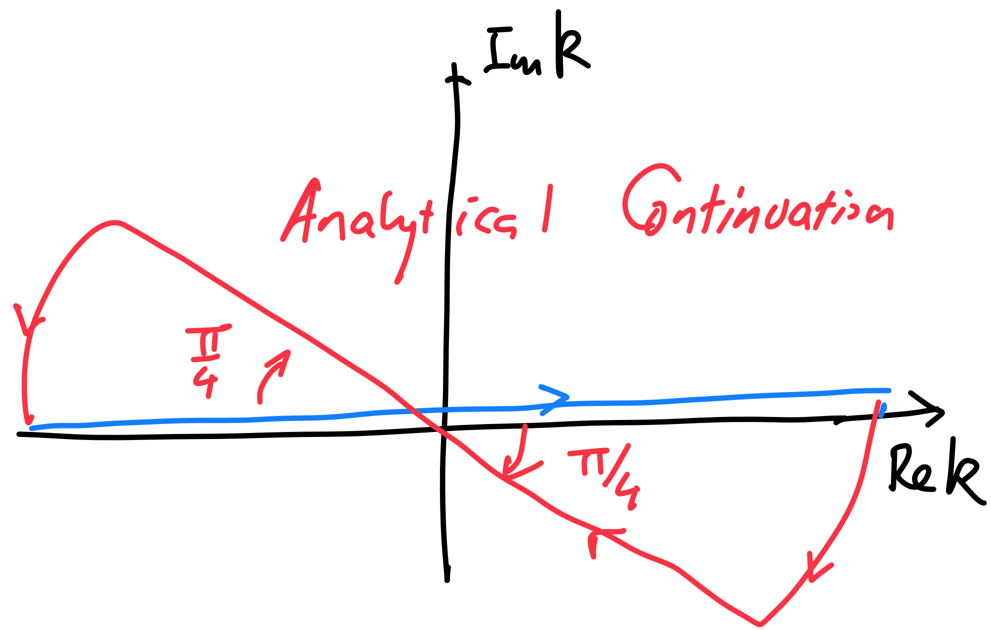
$$\langle 0 | T\, b_{\vec r}(t)\, b_0^\dagger(0) | 0 \rangle \overset{t>0}{=} \sum_{\vec k} \langle 0 | \, b_{\vec r}(t) \underbrace{|\vec k\rangle\langle \vec k|}_{\text{Single particle state}} b_0^\dagger(0) | 0 \rangle$$

$(m = 1)$

$$= \frac{1}{\Omega} \sum_{|\vec k|} e^{i \vec k \cdot (\vec r - 0)} \, e^{-i \varepsilon_k t}$$

$\Omega \to$ vol

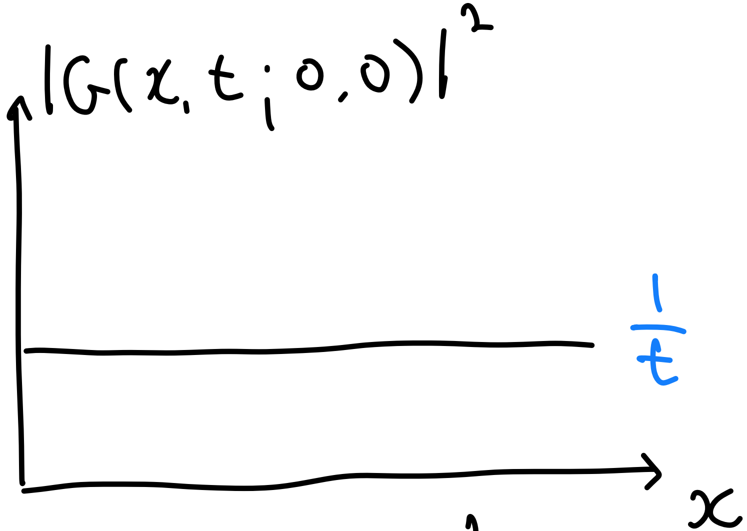
$$\hat{=}_{1D} \; \frac{\hbar^{-\frac{1}{2}}}{\sqrt{2\pi t i}} \, e^{i \frac{x^2}{2t} \cdot \frac{1}{\hbar^{1/2}}}$$

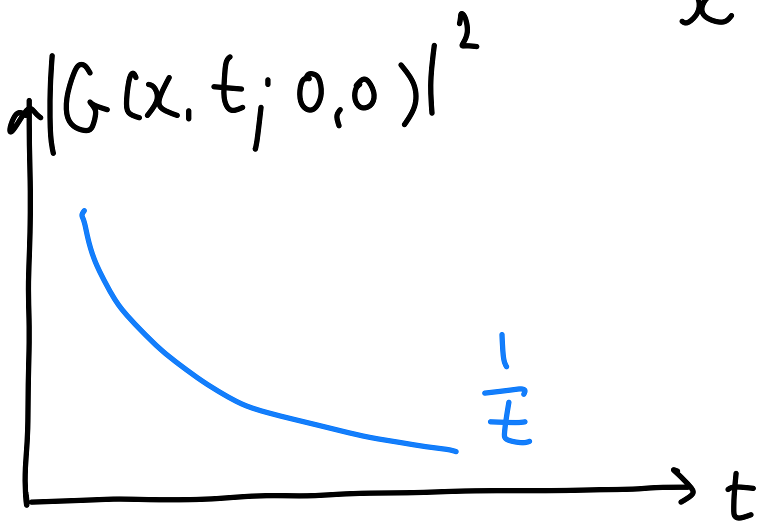


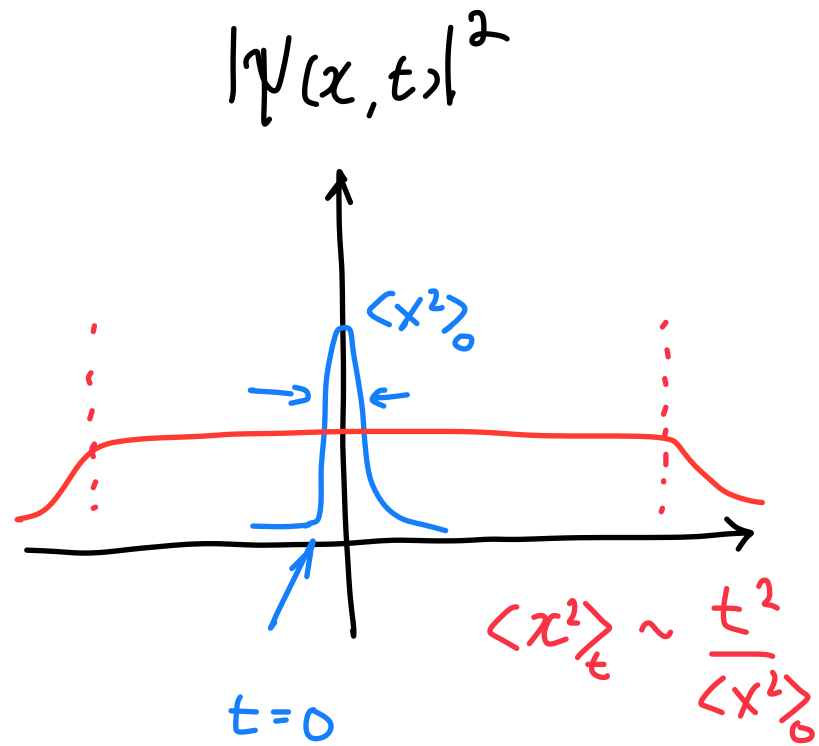
QM 100

O $\qquad$ D

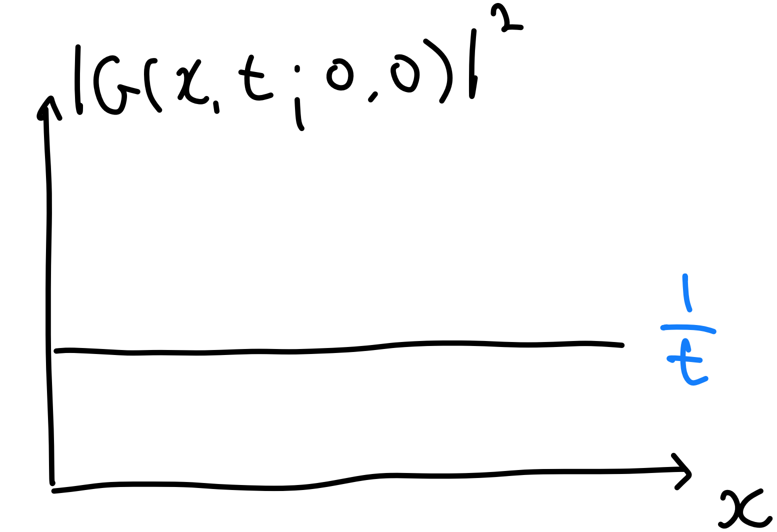
$(0,0)$ $\qquad$ (x,t)

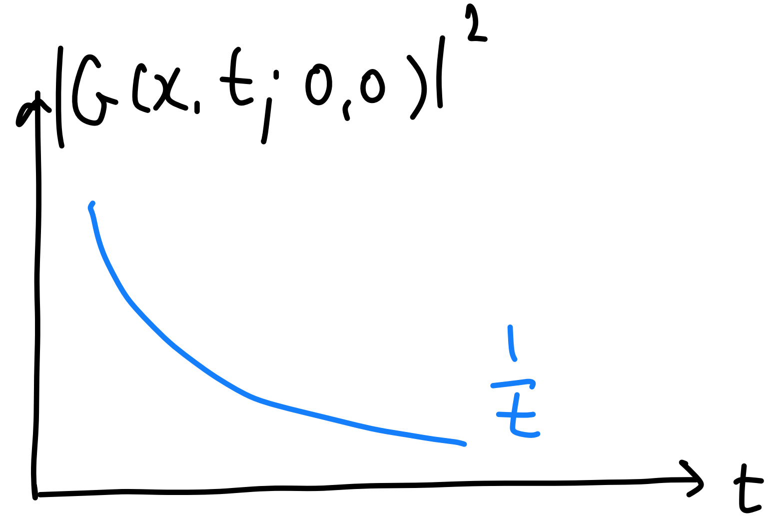




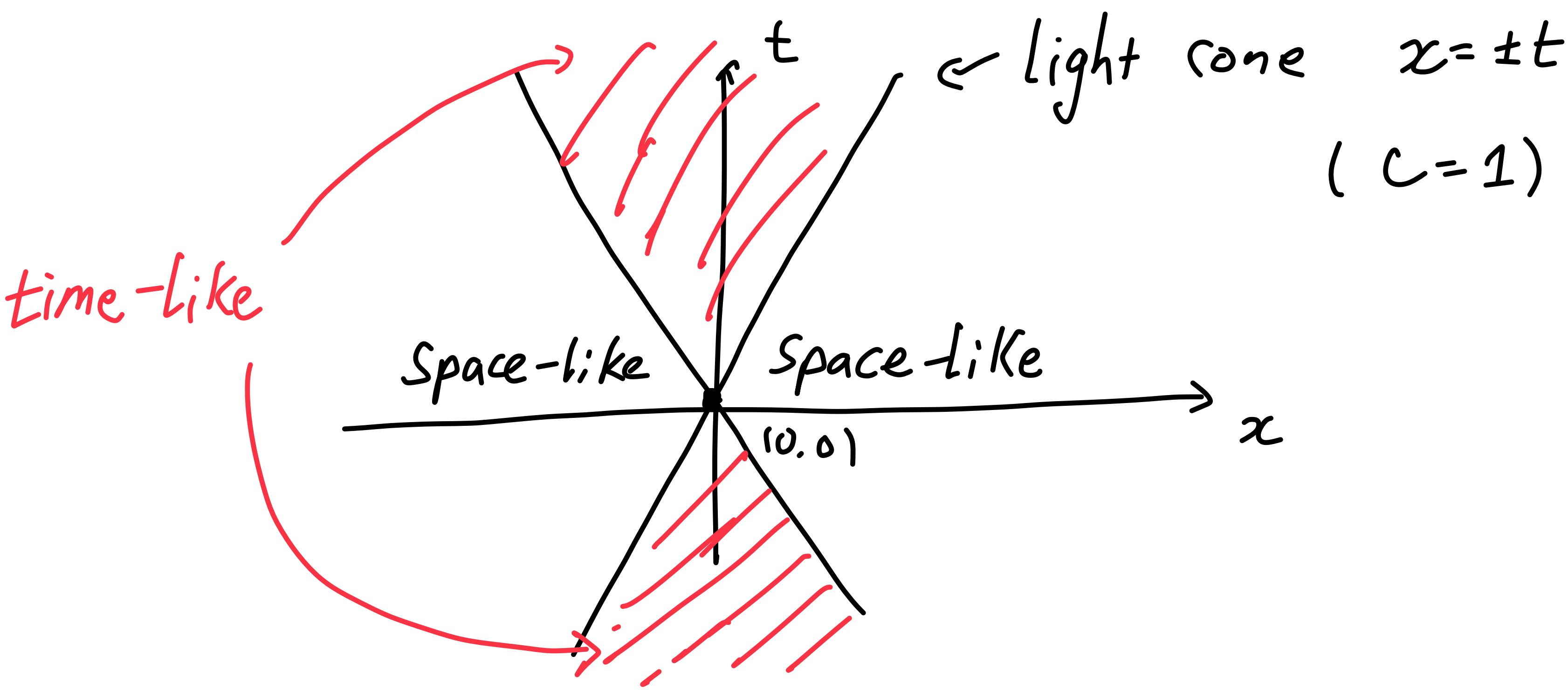


$$|\psi(x,t)|^2 \xrightarrow{\langle x^2\rangle_0 \to 0} |G(x,t;0,0)|^2$$





What about $Z=1$ Relativistic QCP?



$$G_R(\vec{r}, t; 0.0) = \int \frac{1}{(\omega + i\delta)^2 - k^2} e^{i(kx - \omega t)} \frac{d\omega}{2\pi} \frac{dk}{2\pi}$$

$$= \Theta(t) \left[-i \int \frac{e^{ik(x-t)}}{2k} \frac{dk}{2\pi} + i \int \frac{e^{ik(x+t)}}{2k} \frac{dk}{2\pi} \right]$$

$$\frac{1}{2}\Theta(x-t) - \frac{1}{2}\Theta(x+t)$$

$$= \frac{1}{2}\Theta(t)\,\Theta(t - |x|)$$