Phys525:
Quantum Condensed Matter Physics: Quantum Criticality
Basics, Dynamics and Topological criticality

Episode 9:

Imaginary time evolution——slightly more technical stuff

Back to the quantum Ising model.

Two important ideas in QFT/EFT approaches

- Imaginary time evaluation of a quantum problem in d-dimensions=*partition-like functions* in d+1 dimensions (to identify space time symmetries in the QFT representations.)

- Coarse graining approach to lattice Models with discrete or continuous fields (it is used in both classical SM and quantum many-body physics. In many quantum problems, formally and technically constructing via Callan-Symanzik RGE approach).

imaginary time evolution (QFT: Vac-Vac Amplitude)

- Extract ground state properties (today)
- Extract space-time symmetries suitable for EFTs (today)
- Evaluate the dynamic correlations (later discussions)

Quantum states of Ising Model

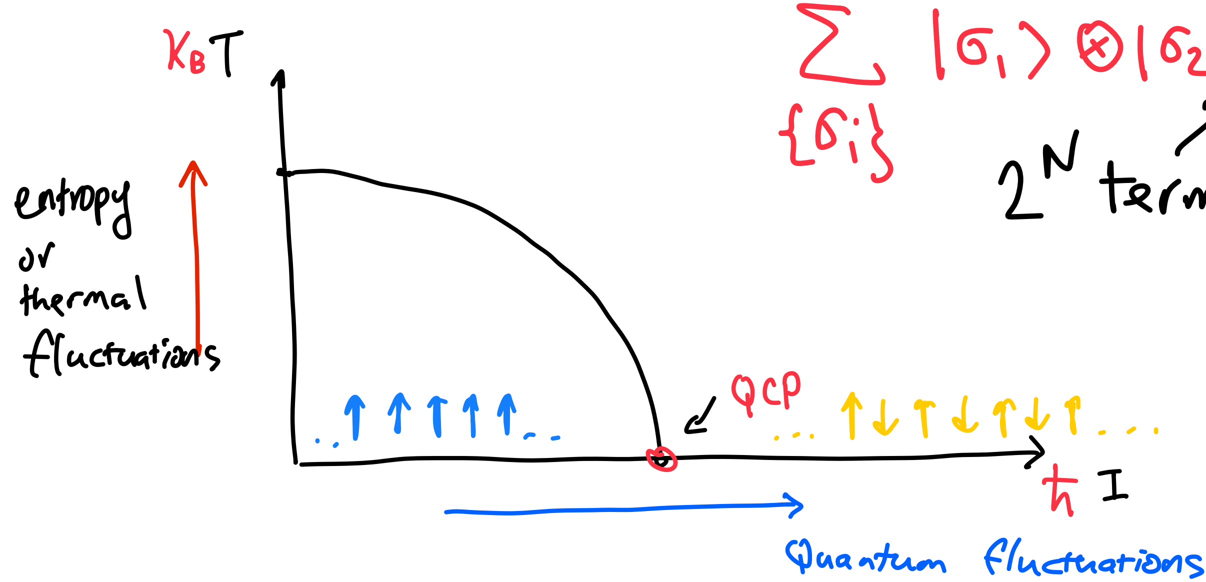
$$|\Psi_{g.s}\rangle = \sum_{\{\sigma_i\}} \Psi(\sigma_1, \sigma_2, \sigma_3, \ldots \sigma_N)\, |\sigma_1\rangle \otimes |\sigma_2\rangle \otimes |\sigma_3\rangle \ldots \otimes |\sigma_N\rangle$$

$$\left\{\begin{array}{l} I=0, \quad \Psi(\sigma_1, \sigma_2, \sigma_3, \ldots \sigma_N) = \underbrace{\delta_{\sigma_1,+1}\, \delta_{\sigma_2,+1}\, \delta_{\sigma_3,+1} \cdots \delta_{\sigma_N,+1}}_{\text{product state in terms of } |\sigma_i\rangle} \\ I=\infty, \quad \Psi(\sigma_1, \sigma_2, \sigma_3, \ldots \sigma_N) = \text{Const} \end{array}\right.$$

"Equal-amplitude spin state"

$$\sum_{\{\sigma_i\}} |\sigma_1\rangle \otimes |\sigma_2\rangle \otimes |\sigma_3\rangle \ldots \otimes |\sigma_N\rangle$$

2^N terms



Some basic Approaches to many-body ground state

Imginary Time evolution

$$Z = \langle -\tfrac{T}{2} | \tfrac{T}{2} \rangle = \langle -\tfrac{T}{2} | e^{-HT} | -\tfrac{T}{2} \rangle$$

$$= \sum_n \langle -\tfrac{T}{2} | n \rangle \langle n | -\tfrac{T}{2} \rangle e^{-E_n \cdot T}$$

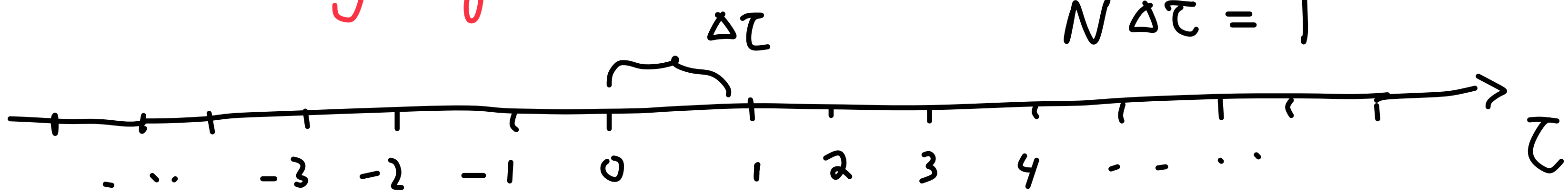
$$\xrightarrow{T \to \infty} |\langle -\tfrac{T}{2} | g.s. \rangle|^2 e^{-E_{g.s.} T}$$

$$E_{g.s.} = \lim_{T \to \infty} \frac{-1}{T} \ln Z(T)$$

Some basic Approaches to many-body ground state

Imaginary Time evolution

$N \Delta\tau = T$



$$Z = \langle 0 | e^{-HT} | 0 \rangle = \langle 0 | e^{-H\Delta\tau} \cdots e^{-H\Delta\tau} | 0 \rangle$$

$$= \cdots \sum_{\{n_0\}} \sum_{\{n_1\}} \sum_{\{n_2\}} \cdots \langle n_0 | e^{-H\Delta\tau} | n_1 \rangle \langle n_1 | e^{-H\Delta\tau} | n_2 \rangle \cdots$$

Complete Set

For Ising Model $\sum_{\{n_0\}} \rightarrow \sum_{\{\sigma_i\},\ i=1\cdots N}$, $|n_1\rangle = \underbrace{|\sigma_1\rangle \otimes |\sigma_2\rangle \otimes \cdots |\sigma_N\rangle}$

2^N states $\leftarrow |\{\sigma_i\},\ i=1\cdots N\rangle$

A technical Remark: $H = H_1 + H_2$, $[H_1, H_2] \neq 0$

$$\langle n_1 | e^{-H\Delta\tau} | n_2 \rangle \overset{\Delta\tau \to 0}{\simeq} \langle n_1 | 1 - (H_1\Delta\tau + H_2\Delta\tau) | n_2 \rangle + O(\Delta\tau^2)$$

$$\simeq \langle n_1 | e^{-H_1\Delta\tau} e^{-H_2\Delta\tau} | n_2 \rangle$$

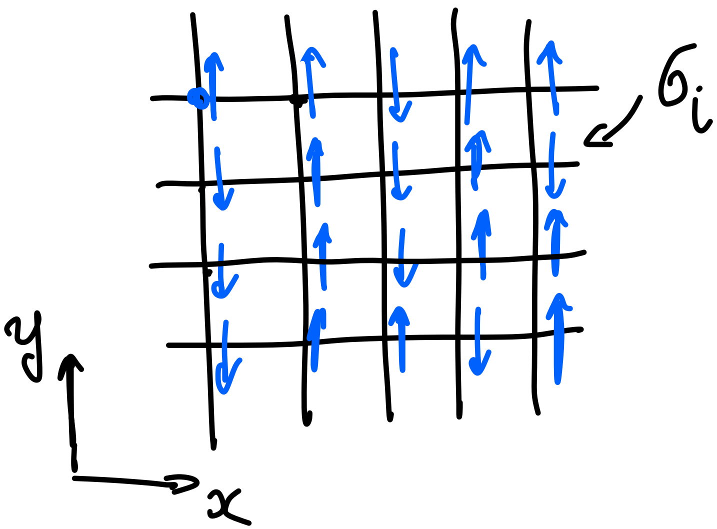
Sometimes

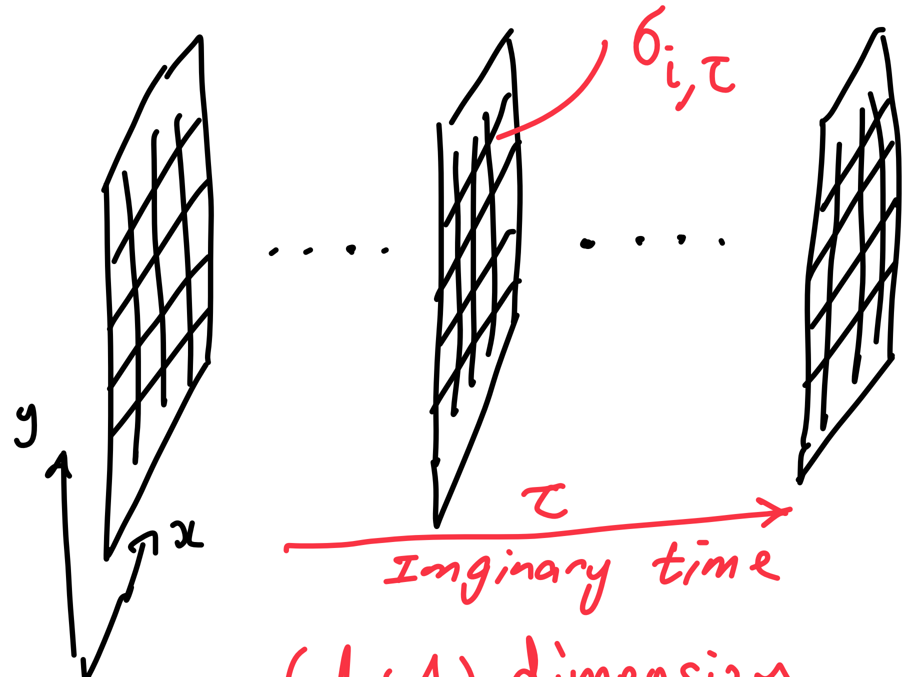
$$\simeq \sum_{\{M\}} \langle n_1 | e^{-H_1\Delta\tau} | M \rangle \langle M | e^{-H_2\Delta\tau} | n_2 \rangle$$

This is needed for Superconducting problem.

Towards Quantum Models   Why QCP ?

$$\mathcal{H}_{Ising} = -J \sum_{\langle ij \rangle} S_{z_i} \cdot S_{z_j} - I \sum_i S_{x_i} \quad \text{(d-dimension)}$$

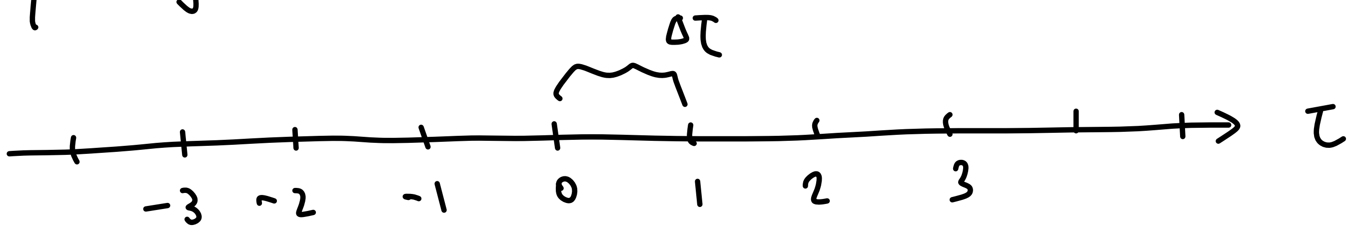




(d+1) dimension

$$\beta \mathcal{H}_{Ising} = -\tilde{J}(J\Delta\tau) \sum_{\langle ij \rangle} \sigma_{i,\tau}\, \sigma_{j,\tau} - \tilde{K}(I\Delta\tau) \sum_{\langle \tau\tau' \rangle} \sigma_{i,\tau} \cdot \sigma_{i,\tau'}$$

One Spin System $\quad H = h\sigma_x$



$$\langle 0| e^{-HT} |0\rangle = \sum_{\cdots\sigma_0=\pm1} \sum_{\sigma_1=\pm1} \sum_{\sigma_2=\pm1} \cdots \langle\sigma_0| e^{-h\sigma_x\Delta\tau} |\sigma_1\rangle\langle\sigma_1| e^{-h\sigma_x\Delta\tau} |\sigma_2\rangle \cdots$$

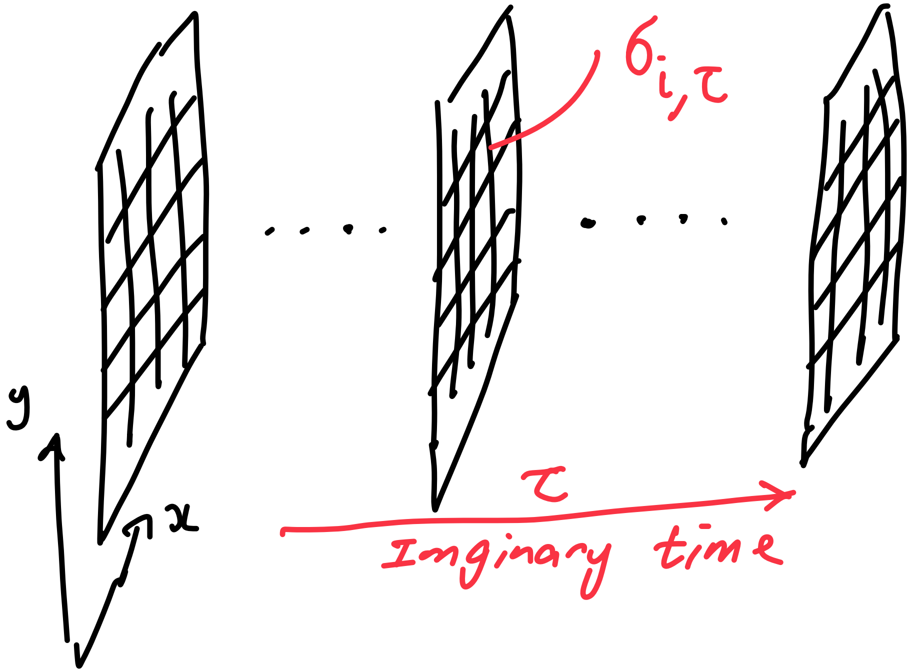
$$\langle\sigma_1| e^{-h\sigma_x\Delta\tau} |\sigma_2\rangle = \cosh h\Delta\tau + \sigma_x \sinh h\Delta\tau$$

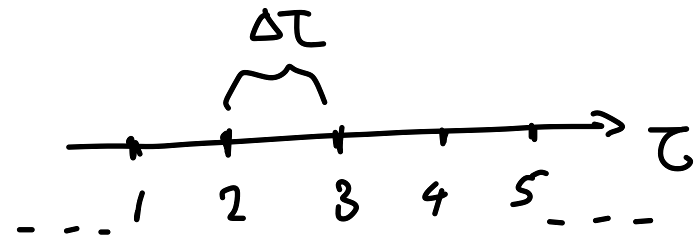
transition Amplitude

$$= X \cdot \underbrace{\begin{bmatrix} e^{\tilde{K}} & e^{-\tilde{K}} \\ e^{-\tilde{K}} & e^{\tilde{K}} \end{bmatrix}}_{e^{\tilde{K}(h\Delta\tau)\sigma_1\sigma_2}}$$

||

Boltzmann Weight →



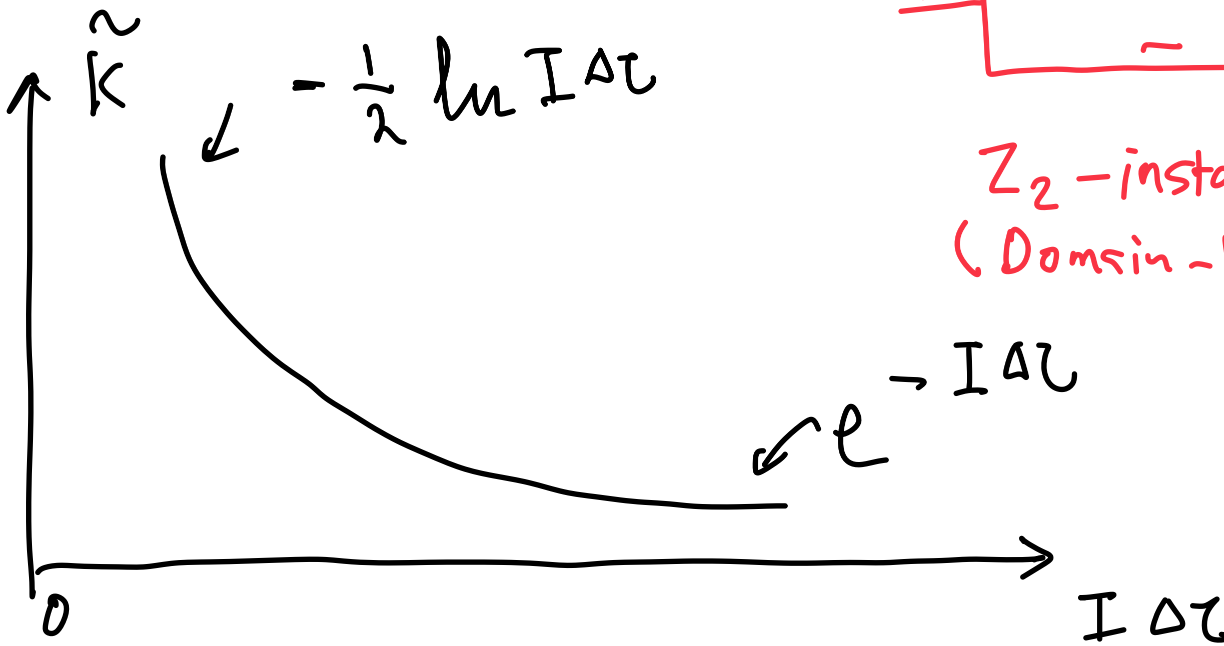


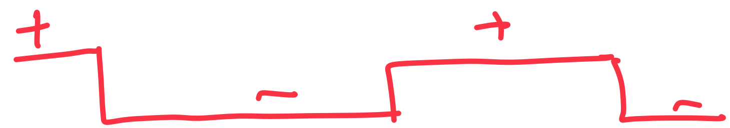
$$\tilde{J}(J\Delta\tau) = J\Delta\tau, \quad \tilde{K}(I\Delta\tau) = -\frac{1}{2}\ln\tanh(I\Delta\tau)$$

$$\beta\mathcal{H}_{\text{Ising}} = -\tilde{J}(J\Delta\tau)\sum_{\langle ij\rangle}\sigma_{i,\tau}\sigma_{j,\tau} - \tilde{K}(I\Delta\tau)\sum_{\langle\tau\tau'\rangle}\sigma_{i,\tau}\cdot\sigma_{i,\tau'}$$

Temporal Coupling due to transverse fields

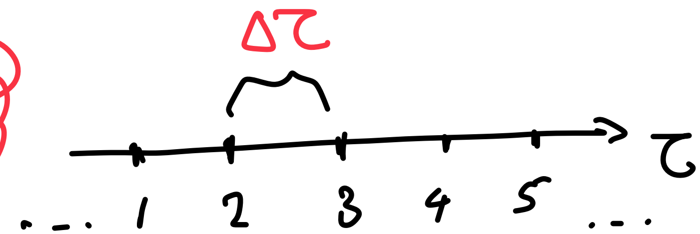
$$\tilde{K}(I\Delta\tau) = -\frac{1}{2}\ln\tanh(I\Delta\tau)$$





Z_2-instanton
(Domain-Walls)

The physics shall not depend on "$\Delta\tau$" : discretization procedure



$I = gJ$, $\tilde{K}(I\Delta\tau) = \tilde{K}(g, J\Delta\tau)$; $\tilde{J} = J\Delta\tau$

Set $\tilde{K} = \tilde{J}$ to deform into <u>isotropic</u> Model

$$\tilde{K}(g, J\Delta\tau) = \tilde{J}(J\Delta\tau) \longrightarrow J\Delta\tau = f(g)$$

$\tilde{K} = \tilde{J} = f(g)$, 1D Ising Model $\longrightarrow f(g) = 1$ $g = g_c$

transition

QCP at $g = g_c$.

$$\beta\mathcal{H}_{Ising} = -\tilde{J}(J\Delta\tau)\sum_{\langle ij\rangle} \sigma_{i,\tau}\sigma_{j,\tau} - \tilde{K}(I\Delta\tau)\sum_{\langle\tau\tau'\rangle} \sigma_{i,\tau}\cdot\sigma_{i,\tau'}$$