Phys525:
Quantum Condensed Matter Physics:
emergent symmetry and phenomena

Free fermion Topological states:

examples and classifications: back to non-interacting systems

"Topological Order"

Wilson, 74

$W_c = e^{i\oint_c \vec{A}\cdot d\vec{x}}$

Non-Local Order

$S_{VN} \simeq \alpha L - \gamma_{Topo}$

FQHE → Resonance Valence bond

(Wen)

(QSL)

Strongly interacting

(Mainly in 2D)

"Topological States"

Topological invariants

Winding Numbers

(Related to Chern Numb)

Valence-bond-Solid ← QHE

(VBS)

(TKNN 82)

free fermions

(Global Symmetries)

interacting/Non-interacting

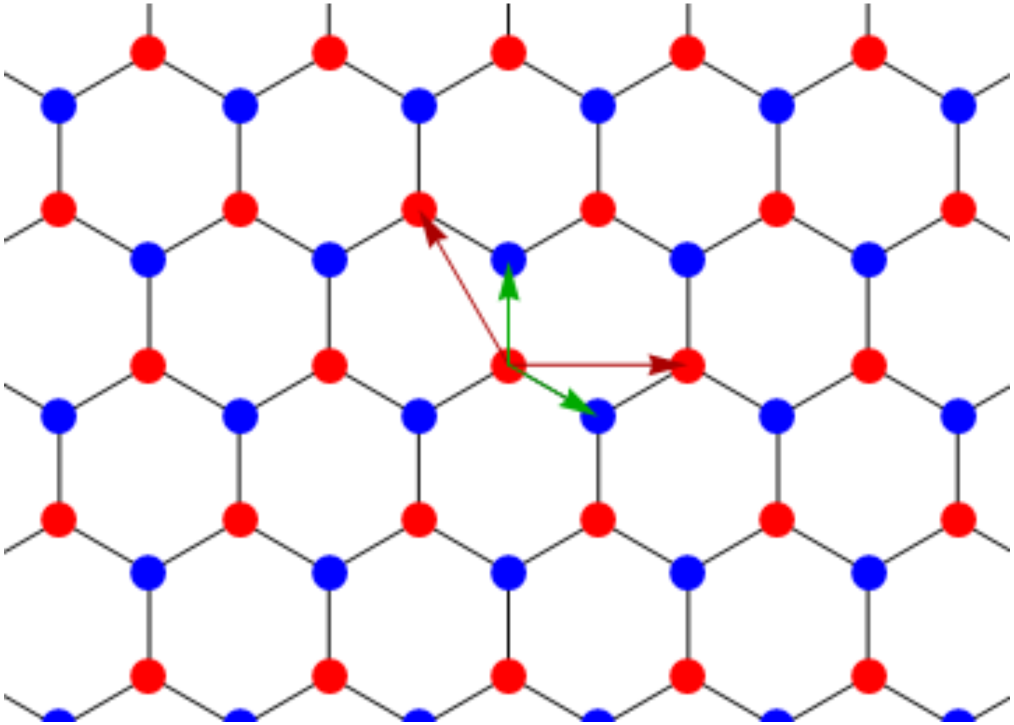
(1D, 2D, 3D)

- From the Haldane model of the Chern-insulator (see L2 as well) (1988) to Z_2 topological insulator by Kane-Mele (2005) and Fu-Kane-Mele theory.

- Ten-fold-way classification via Atland-Zirnbauer RMT Hamiltonian classes (Schnyder, Ryu, Frusaki, Ludwig, 2008; also Kitaev, 2009).

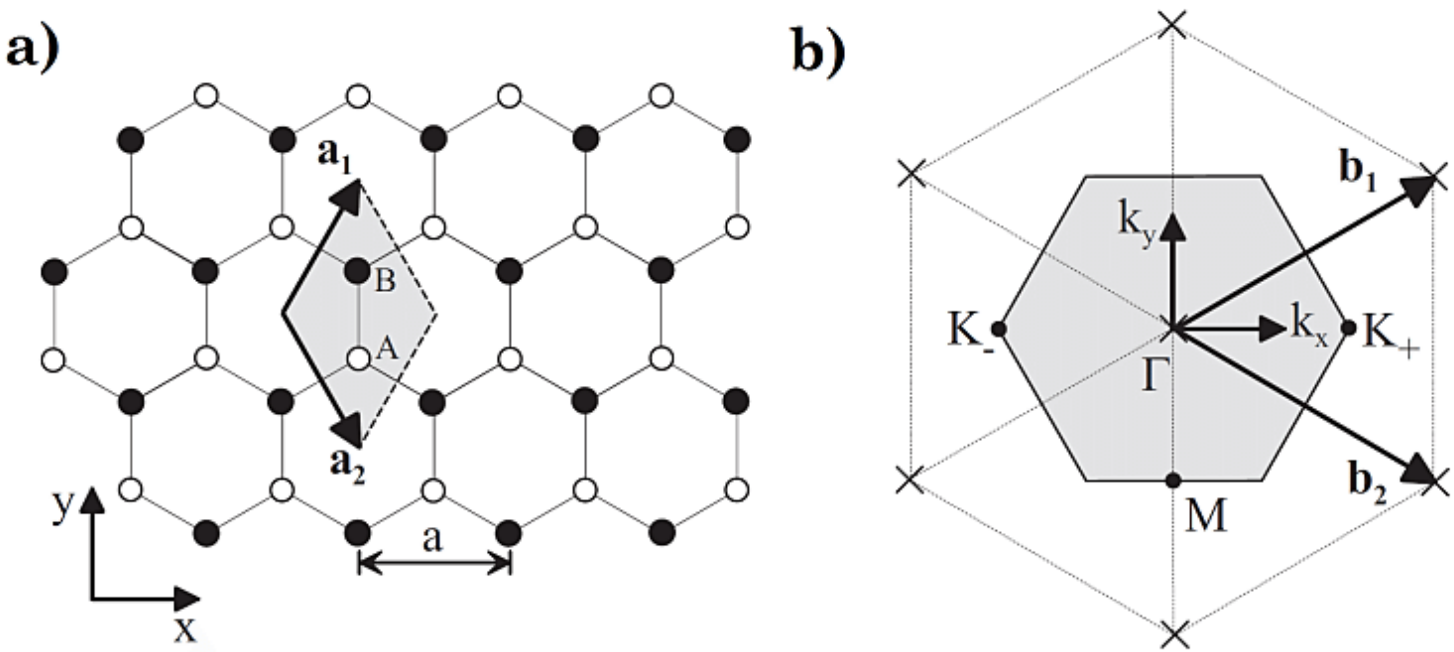
- Topological superconductors with time reversal symmetry (symmetry protected) and Fermi surface classifications (by Qi-Hughes-Zhang, 2010).

Discussions on a more realistic Haldane model in honeycomb lattices



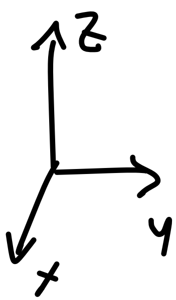
$$\mathcal{H}(p, m) = \tau_z \otimes \sigma_x p_x + \sigma_y p_y + m\sigma_z + m_H \tau_z \otimes \sigma_z$$

τ_z : Valley Subspace ; $\sigma_{x,y,z}$: Sublattice Subspace



Edge States

$$H = \tau_z \otimes \sigma_x p_x + \sigma_y p_y + \boxed{m\,\sigma_z}$$ TRI



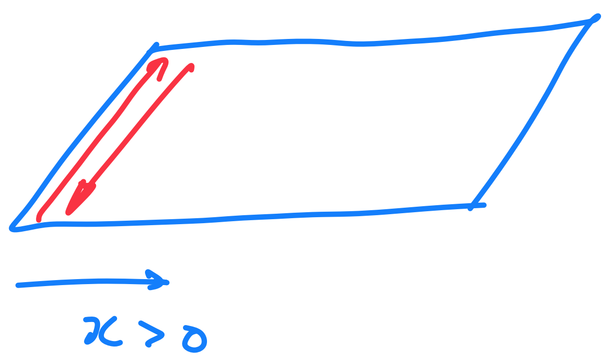
$$\left[\to \; H = \tau_z \otimes \sigma_y p_x + m\sigma_x + \sigma_z p_y\right]$$

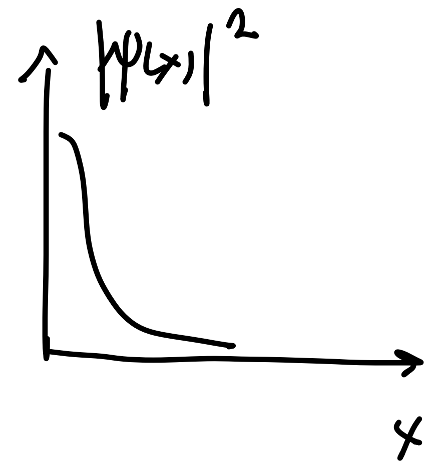
$\tau_z = \pm 1$

$$\begin{bmatrix} p_y & \mp i p_x + m \\ \pm i p_x + m & -p_y \end{bmatrix} |\psi_{edge}\rangle = \pm p_y |\psi_{edge}\rangle$$

$$|\psi_{edge}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{i p_y \cdot y - m x} \quad \text{or} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{i p_y y - m x}$$

$$\mathcal{E}(p_y) = \pm p_y v$$



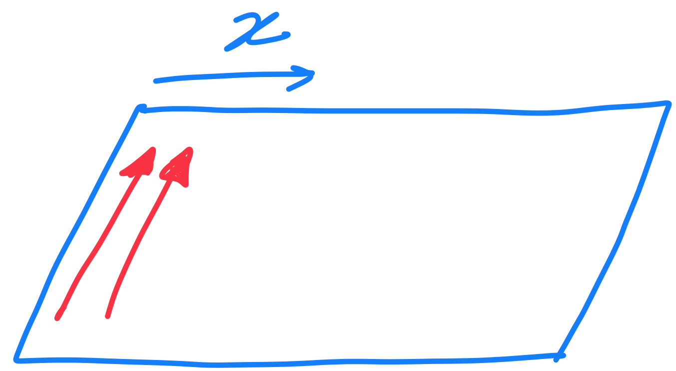


$$H = \tau_z \otimes (\sigma_x p_x + \boxed{m_H \sigma_z}) + \sigma_y p_y$$

$$\rightarrow \quad H = \tau_z \otimes (\sigma_y p_x + m_H \sigma_x) + \sigma_z p_y$$

$$|\psi_{edge}\rangle = e^{i p_y y - m_H x} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ for both $\tau_z = \pm 1$

$$\mathcal{E}(p_y) = p_y \cdot v$$

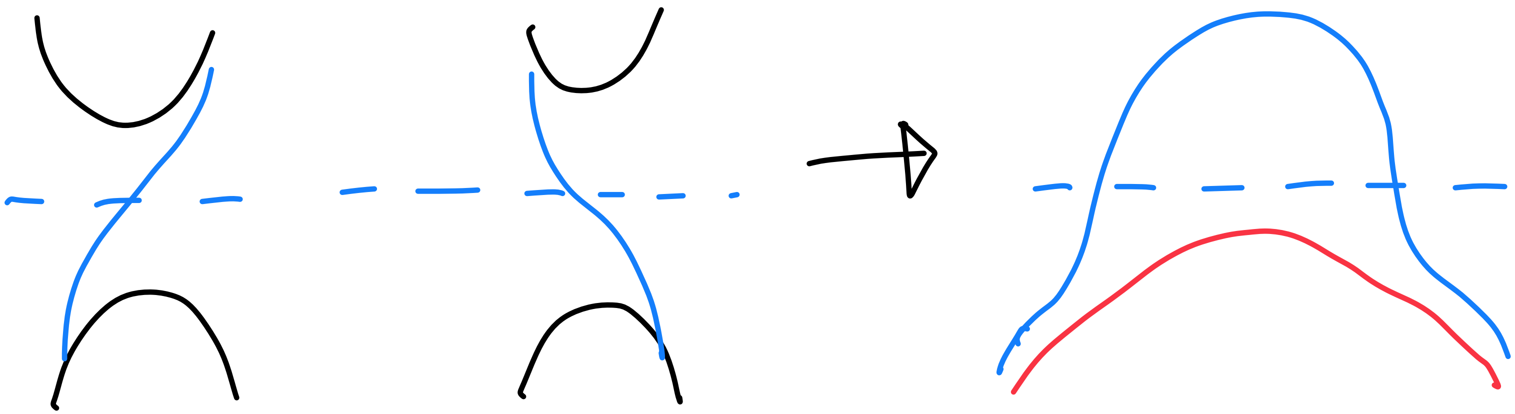


Taking into PHC, $\tau_z = 1$ $\quad \tau_z = -1$

$$\sigma_H = \frac{1}{2}\left[\text{Sign}\, m_+ - \text{Sign}\, m_-\right] = 0, \pm 1$$

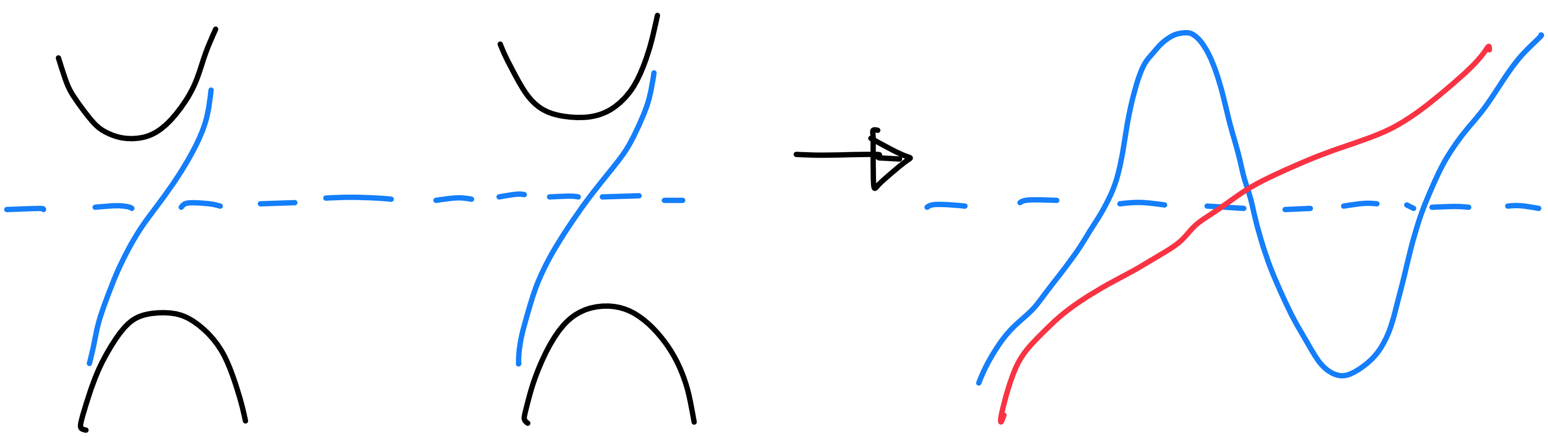
A) $m_H = 0$, $m \neq 0$

$m_+ \cdot m_- > 0$



B) $m_H \neq 0$, $m = 0$

$m_+ \cdot m_- < 0$



Thouless-Komoto-Nightingale-den Nijs 1982

$$C = \frac{1}{2\pi}\int_{\text{band}} \left\{ \partial_{k_x} A_y - \partial_{k_y} A_x \right\} \underline{dk_x dk_y}$$

$$\vec{A}(\vec{k}) = i \langle u_k | \vec{\nabla}_k | u_k \rangle \quad (\text{1-form})$$

$$\begin{cases} = \frac{i}{2}\left\{ \langle u_k | \vec{\nabla}_k | u_k \rangle - \langle \vec{\nabla}_k u_k | u_k \rangle \right\} \\ = -\operatorname{Im} \langle u_k | \vec{\nabla}_k | u_k \rangle \end{cases}$$

Alternatively

(This is the first class Chern-Number.)

2nd Class Chern Number $(d = (4+1)D)$

$$A^{\alpha\beta}_{k} = i \langle \alpha, k | \nabla_k | \beta, k \rangle, \quad \alpha, \beta = \text{band indices}$$

$$C_2 = \frac{1}{32\pi^2} \int d^4\vec{k}\ \varepsilon_{ijkl} \, Tr\{F_{ij} F_{jl}\}$$

$$F^{\alpha\beta}_{ij} = \partial_i A^{\alpha\beta}_{j} - \partial_j A^{\alpha\beta}_{i} - [A_i, A_j]^{\alpha\beta}$$

(Similar index for Polyakov-t'Hooft Monopole in Yang-Mills Fields / QFT.)

Kane - Mele Model (Spin - 1/2)

$$H = \tau_z \otimes \sigma_x p_x + \sigma_y p_y + m_{KM} \underbrace{\tau_z \otimes \sigma_z \otimes s_z}_{TRI}$$

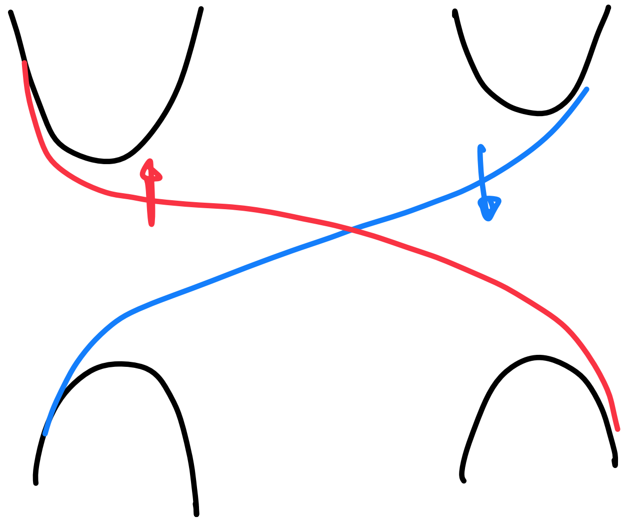
$\sigma_z \otimes \tau_z$ parity even, TR odd;

$\tau_z \otimes s_z \otimes \sigma_z$ parity even, TR even;

σ_z parity odd;

τ_z parity odd, TR odd;

s_z parity even, TR odd;



Kramer doublets

Two copies of Haldane states

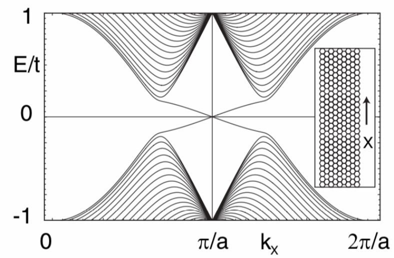


FIG. 1. (a) One-dimensional energy bands for a strip of graphene (shown in inset) modeled by (7) with $t_2/t = 0.03$. The bands crossing the gap are spin filtered edge states.

of K and K' are both at $k = 0$. It is interesting to note that for zigzag edges the edge states persist for $\Delta_{\rm so} \to 0$, where they become perfectly flat [16]. This leads to an enhanced density of states at the Fermi energy associated with zigzag edges. This has been recently seen in scanning tunneling spectroscopy of graphite surfaces [17].

We have also considered a nearest neighbor Rashba term, of the form $i\hat{z} \cdot (\mathbf{s}_{\alpha\beta} \times \mathbf{d}) c^\dagger_{i\alpha} c_{j\beta}$. This violates the conservation of s_z, so that the Laughlin argument no longer applies. Nonetheless, we find that the gapless edge states remain, provided $\lambda_R < \Delta_{\rm so}$, so that the bulk band gap remains intact. The crossing of the edge states at the Brillouin zone boundary $k_x = \pi/a$ in Fig. 1 (or at $k = 0$ for the armchair edge) is protected by time reversal symmetry. The two states at $k_x = \pi/a$ form a Kramers doublet whose degeneracy cannot be lifted by any time reversal symmetric perturbation. Moreover, the degenerate states at $k_x = \pi/a \pm q$ are a Kramers doublet. This means that elastic backscattering from a random potential is forbidden. More generally, scattering from a region of disorder can be characterized by a 2×2 unitary S matrix which relates the incoming and outgoing states: $\Phi_{\rm out} = S\Phi_{\rm in}$, where Φ is a two component spinor consisting of the left and right moving edge states $\phi_{L\uparrow}$, $\phi_{R\downarrow}$. Under time reversal $\Phi_{\rm in,out} \to s_y \Phi^*_{\rm out,in}$. Time reversal symmetry therefore imposes the constraint $S = s_y S^T s_y$, which rules out any off diagonal elements.

Electron interactions can lead to backscattering. For instance, the term $u\psi^\dagger_{L\uparrow}\partial_x\psi^\dagger_{L\uparrow}\psi_{R\downarrow}\partial_x\psi_{R\downarrow}$, does not violate time reversal, and will be present in an interacting Hamiltonian. For weak interactions this term is *irrelevant* under the renormalization group, since its scaling dimension is $\Delta = 4$. It thus will not lead to an energy gap or to localization. Nonetheless, it allows inelastic backscattering. To leading order in u it gives a finite *conductivity* of the edge states, which diverges at low temperature as $u^{-2}T^{3-2\Delta}$ [18]. Since elastic backscattering is prevented by time reversal there are no relevant backscattering processes for weak interactions. This stability against interactions and disorder distinguishes the spin filtered edge states from ordinary one-dimensional wires, which are localized by weak disorder.

A parallel magnetic field $H_\parallel$ breaks time reversal and leads to an avoided crossing of the edge states. $H_\parallel$ also reduces the symmetry, allowing terms in the Hamiltonian which provide a continuously gapped path connecting the states generated by $\sigma_z\tau_z s_z$ and σ_z. Thus in addition to gapping the edge states $H_\parallel$ eliminates the topological distinction between the QSH phase and a simple insulator.

The spin filtered edge states have important consequences for both the transport of charge and spin. In the limit of low temperature we may ignore the inelastic backscattering processes, and describe the ballistic transport in the edge states within a Landauer-Büttiker [19] framework. For a two terminal geometry [Fig. 2(a)], we predict a ballistic two terminal charge conductance $G = 2e^2/h$. For the spin filtered edge states the edge current density is related to the spin density, since both depend on $n_{R\uparrow} - n_{L\downarrow}$. Thus the charge current is accompanied by spin accumulation at the edges. The interplay between charge and spin can be probed in a multiterminal device. Define the multiterminal spin conductance by $I^s_i = \sum_j G^s_{ij} V_j$. Time reversal symmetry requires $G^s_{ji} = -G^s_{ij}$, and from Fig. 2(b) it is clear that $G^s_{ij} = \pm e/4\pi$ for adjacent contacts i and j. In the four terminal geometry of Fig. 2(b) a spin current $I^s = eV/4\pi$ flows into the right contact. This geometry can also be used to *measure* a spin current. A spin current incident from the left (injected, for instance,

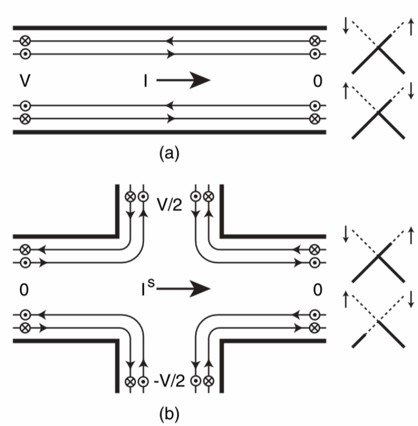


FIG. 2. Schematic diagrams showing (a) two terminal and (b) four terminal measurement geometries. In (a) a charge current $I = (2e^2/h)$ V flows into the right lead. In (b) a spin current $I^s = (e/4\pi)$ V flows into the right lead. The diagrams to the right indicate the population of the edge states.

Ten-fold way classification

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0803.2786 Schnyder et Al. On ten-fold way symmetry classes

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		TRS	PHS	SLS	$d=1$	$d=2$	$d=3$
standard (Wigner-Dyson)	A (unitary)	0	0	0	-	$\mathbb{Z}$	-
	AI (orthogonal)	+1	0	0	-	-	-
	AII (symplectic)	-1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$
chiral (sublattice)	AIII (chiral unitary)	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$
	BDI (chiral orthogonal)	+1	+1	1	$\mathbb{Z}$	-	-
	CII (chiral symplectic)	-1	-1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$
BdG	D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-
	C	0	-1	0	-	$\mathbb{Z}$	-
	DIII	-1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	CI	+1	-1	1	-	-	$\mathbb{Z}$

'ABLE I: Ten symmetry classes of single particle Hamiltonians classified in terms of the presence or absence of time-reversal ymmetry (TRS) and particle-hole symmetry (PHS), as well as sublattice (or "chiral") symmetry (SLS).[36,37] In the table, the bsence of symmetries is denoted by "0". The presence of these symmetries is denoted either by "+1" or "−1", depending n whether the (antiunitary) operator implementing the symmetry at the level of the single-particle Hamiltonian squares to +1" or "−1" (see text). [The index ±1 equals η_c in Eq. (1b); here $\epsilon_c = +1, -1$ for TRS and PHS, respectively.] For the first ix entries of the TABLE (which can be realized in non-superconducting systems) TRS = +1 when the SU(2) spin is integer called TRS (even) in the text] and TRS = −1 when it is a half-integer [called TRS (odd) in the text]. For the last four ntries, the superconductor "Bogoliubov-de Gennes" (BdG) symmetry classes D, C, DIII, and CI, the Hamiltonian preserves U(2) spin-1/2 rotation symmetry when PHS=−1 [called PHS (singlet) in the text], while it does not preserve SU(2) when 'HS=+1 [called PHS (triplet) in the text]. The last three columns list all topologically non-trivial quantum ground states as function of symmetry class and spatial dimension. The symbols $\mathbb{Z}$ and $\mathbb{Z}_2$ indicate whether the space of quantum ground tates is partitioned into topological sectors labeled by an integer or a $\mathbb{Z}_2$ quantity, respectively.

legenerate band crossings (Dirac points) in the spectrum topological phases, we shall be mostly concerned with

AZ class	SU(2)	TRS	Constraints on Hamiltonians	Examples in 2D
D	×	×	$t_x\mathcal{H}^T t_x = -\mathcal{H}$	Spinless chiral $(p \pm ip)$-wave
DIII	×	◯	$t_x\mathcal{H}^T t_x = -\mathcal{H}$, $is_y\mathcal{H}^T(-is_y) = \mathcal{H}$	Superposition of $(p+ip)$- and $(p-ip)$-wave
A	△	×	no constraint	Spinful chiral $(p \pm ip)$-wave
AIII	△	◯	$r_y\mathcal{H}r_y = -\mathcal{H}$	Spinful p_x- or p_y-wave
C	◯	×	$r_y\mathcal{H}^T r_y = -\mathcal{H}$	$(d \pm id)$-wave
CI	◯	◯	$r_y\mathcal{H}^T r_y = -\mathcal{H}$, $\mathcal{H}^* = \mathcal{H}$	$d_{x^2-y^2}$- or d_{xy}-wave

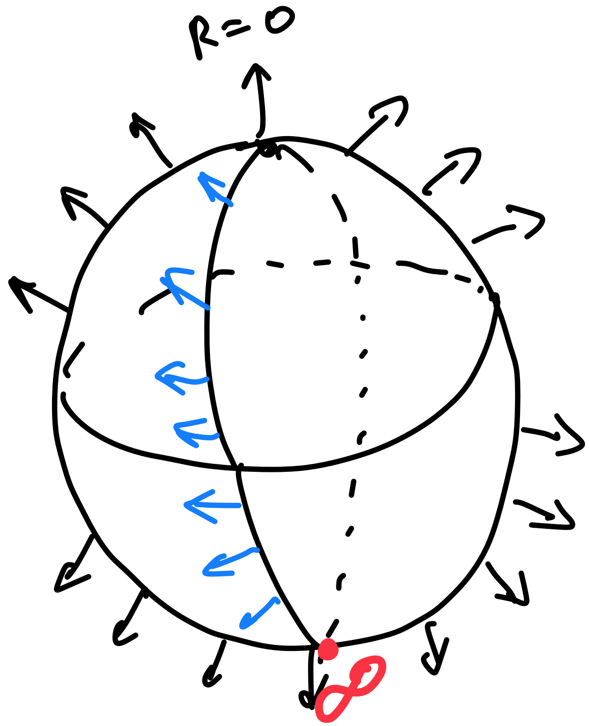
TABLE II: Symmetry classification of Bogoliubov-de Gennes systems, in terms of the presence ("◯") or absence ("×") of SU(2) spin rotation symmetry and time-reversal symmetry (TRS). In classes A and AIII, Hamiltonians are invariant under rotations about the z- (or any fixed) axis in spin space, but not under full SU(2) rotations, as denoted by "△" in the table. The sets of standard Pauli matrices $s_{x,y,z}$, $t_{x,y,z}$, and $r_{x,y,z}$ act on the spin, the particle-hole, and the grading defined in Eq. (16), respectively.

energy dispersion of a single particle. The Hamiltonian has PHS (triplet) [Eq. (7)], $h(k) = -t_x h^T(-k) t_x$.

Class DIII Consider class DIII, which satisfies conditions (a) and (b). A set of matrices which simultaneously satisfy (a) and (b) does not form a subalgebra of so$(4m)$, but consists of all those elements of the Lie algebra so$(4m)$ which are not elements of the sub Lie algebra u$(2m)$.[37]

Combining (a) and (b), one can see that a member of class DIII anticommutes with the unitary matrix $t_x \otimes s_y$,

It is interesting to note, that Hamiltonian (13) is a direct product of Hamiltonian (10), $h(k_x, k_y)$, and $h(-k_x, k_y)$. This follows from a simple reordering of the basis elements in Eq. (13), such that $(\boldsymbol{c}_k^\dagger, \boldsymbol{c}_{-k}) \to (c_{k\uparrow}^\dagger, c_{-k\uparrow}, c_{k\downarrow}^\dagger, c_{-k\downarrow})$. The superconductor described by the order parameter Eq. (13b) can be thought of as a two-dimensional analogue of the BW state realized in the B phase of ^{3}He. The BW state, which is also a member of class DIII and described by the d-vector $\boldsymbol{d}_k = \bar{\Delta}\left(\, k_x,\; k_y,\; k_z\,\right)$, will be discussed in Sec. V as an



$H(\vec{k}): S^2 \to S^2$

N_w = Winding Number

Interactions: $G(i\omega, \vec{k}) \in GL(2, \mathbb{C})$,

2x2 Complex Matrices

$$G(i\omega, \vec{K}) = \frac{1}{i\omega - H_0(\vec{k}) - \Sigma(\vec{k}, \omega)},$$

Space-time structure, $\pi_{d+1}(GL(2,\mathbb{C})) \simeq \pi_{d+1}(U(2))$

$\hat{=} \pi_{d+1}(SU(2)) \hat{=} \pi_{d+1}(S^3)$

$d=2$

$$\pi_{d+1}(SU(2)) = \pi_3(SU(2)) = \pi_3(S^3) = \mathbb{Z}_N$$

$SU(2)$ $$e^{i\vec{\sigma}\cdot\vec{n}\frac{\chi}{2}} = \cos\frac{\chi}{2} + i\,\vec{\sigma}\cdot\vec{n}\,\sin\frac{\chi}{2}$$

$$\chi \in [0, 2\pi], \quad \vec{n} = (\theta, \varphi)$$

$$U(\vec{n}, \chi) = U(\underbrace{\theta}_{[0,\pi]}, \frac{\chi}{2}, \underbrace{\varphi}_{[0,2\pi]}) \in S^3$$

Rep. quaternions: $a + ib + jc + kd$,

$$(i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ji = k)$$

Supplementary Material on $SU(n)$

isomorphism : $U(N) = \dfrac{SU(N) \times U(1)}{Z_n}$

Z_n – Roots of Unity. $e^{i\frac{n}{N}2\pi}$, $n = 0, 1, \cdots N-1$.

homomorphism : $\pi_m(U(N)) = \pi_m(SU(N) \times U(1))$

$u(N) = su(N) \times u(1)$ Group algebra isomorphism

Centre of $SU(N)$: $e^{i\frac{n}{N}n\pi}\mathbb{1}$, $n = 0, \cdots N-1$.

(Commute with every $SU(N)$ element)

N_w for $GL(M, \mathbb{C})$

ferent topological classes across μ_c and therefore can be applied to study transitions involved changes of discrete topologies. This is a very unique feature of topological quantum criticality. That is, effective quantum fields for dynamics at TQCPs *are representations of a topological group so to reflect changes of topologies of underlying many-body states.*

Simplest applications can be easily found in $2d$ time-reversal-symmetry breaking superconductors[7]. In this case, one finds that quantum fields for a TQCP form a nontrivial representation of $\pi_3(\mathcal{G}L(2,\mathbb{C})) = \mathbb{Z}$. At a surface of a fixed chemical potential $\mu(\neq \mu_c)$, the topological invariant can be defined as $(k_0 = \Omega)$

$$N_W = \frac{1}{24\pi^2}\int dk_x dk_y dk_0 \epsilon_{\alpha\beta\gamma}$$
$$Tr G\frac{\partial}{\partial k_\alpha}G^{-1}G\frac{\partial}{\partial k_\beta}G^{-1}G\frac{\partial}{\partial k_\gamma}G^{-1}, \alpha,\beta = 0, x, y. \tag{1}$$

This characterization can be extended to other even-spatial dimensions.

N_W above can also be equivalently expressed in terms of the zero frequency limit of the Green's function, $G(0,\mathbf{k};\mu)$ without frequency integration. Just as in the limit of mean-field or one-particle theory, one can then completely isolate the frequency integration from the momentum summation. The topological invariants above can then be more conveniently expressed in terms of $G_0^{-1}(\mathbf{k},\mu) = G^{-1}(0,\mathbf{k};\mu)$. It is important to notice that although for a given Ω, $G(i\Omega,\mathbf{k};\mu \neq \mu_c)$ are complex $M \times M$ matrices and are non-Hermitian, $G_0(\mathbf{k},\mu)$ are Hermitian following the following general relation

function, $G^{-1}(0,\mathbf{k};\mu)^{-1} = G_0^{-1}(\mathbf{k};\mu)$ that takes into account interaction renormalization effects. Now to apply to d-dimensional spatial space of bulk states instead of $(d+1)$-dimensional space-time , we can again extend the space to $d+1$ dimension by inclusion of an additional dimension of the general tuning parameter μ along which a TQCP occurs. $G_0(\mathbf{k};\mu)$ defined in $(d+1)$ dimension hyperspace, has well defined analytical structures when $\mu \neq \mu_c$ because all fermion fields are well gapped except at $\mu = \mu_c$ or TQCPs; they are also invertible when $\mu \neq \mu_c$ which is the situation we intend to focus on.

At a d-dimension surface of such a $(d+1)$ dimension hyperspace that again excludes enclosed TQCPs (see Fig.1), $G_0^{-1}(\mathbf{k};\mu \neq \mu_c)$ is everywhere smooth and can be thought as a mapping from d-dimension surfaces to a Hamiltonian manifold. This allows a classification of quantum fields using a d-dimensional sphere embedded in $(d+1)$-dimension hyperspace. In the presence of time reversal symmetry in superconductors and therefore a chiral symmetry, $G_0^{-1}(\mathbf{k};\mu)$ can always be cast into an off-diagonal block matrix, with block matrices being $M \times M$ invertible complex matrices, $Q(\mathbf{k})$ and $Q^\dagger(\mathbf{k})$. That is

$$G_0^{-1}(\mathbf{k};\mu) = \begin{bmatrix} 0 & Q(\mathbf{k}) \\ Q^\dagger(\mathbf{k}) & 0 \end{bmatrix}. \tag{3}$$

This structure naturally appears in non-interacting or mean field theories and can be shown to be true in the presence of interactions as far as the chiral symmetry is present[6, 8–11].

N_W for $G(\omega=0, K)$ with T, C. Symmetry

$$N_W = \frac{1}{24\pi^2}\int dk_x dk_y dk_z \epsilon_{\alpha\beta\gamma}$$
$$Tr\Sigma G_0 \frac{\partial}{\partial k_\alpha} G_0^{-1} G_0 \frac{\partial}{\partial k_\beta} G_0^{-1} G_0 \frac{\partial}{\partial k_\gamma} G_0^{-1}, \alpha, \beta = x, y, z. \tag{4}$$

$\Sigma = \mathcal{TC}$ is defined as a unitary chiral transformation anti-commuting with $G_0^{-1}(\mathbf{k};\mu)$ that now is hermitian. The winding number defined in Eq.(1) would vanish if Σ were replaced with a unity matrix and hence it is crucial to have a chiral transformation Σ in its definition.

$$\Sigma^{-1}G_0^{-1}(\mathbf{k})\Sigma = -G_0^{-1}(\mathbf{k}).$$
$$\mathcal{T}^{-1}G_0^{-1}(\mathbf{k})\mathcal{T} = G_0^{-1}(-\mathbf{k}), \mathcal{C}^{-1}G_0^{-1}(\mathbf{k})\mathcal{C} = -G_0^{-1}(-\mathbf{k}). \tag{5}$$

$\mathcal{T}$, $\mathcal{C}$ are anti-unitary time reversal and charge conjugation transformation respectively. In the case of $3d$ p-wave spin triplet superconductors, the effective Hamiltonian manifold H_M suggested by hermitian matrix $G_0^{-1}(\mathbf{k};\mu)$ can be defined on a three sphere S^3. So N_W in Eq.(4) also equivalently represents $\pi_{d=3}(S^3) = \mathbb{Z}$.

For gapless topological phases, topological invariants are defined in an embedded subspace or sphere with dimension lower than the physical space-time or space dimensions. A TQPC connects two states with different topologies in an embedded subsphere and again it crucially depends on the global time reversal symmetry.

For time reversal symmetry breaking states, the embedded space shall be a $[(d-p-1)+1]$ dimension space-time sphere transverse to a Fermi surface of dimension p,

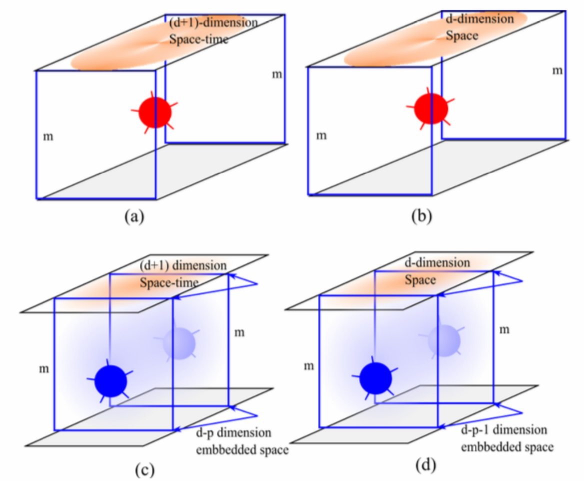


Figure 1. Standard cartoon pictures of changes of global topologies (of integer groups); vertical direction represents a dimension of general tuning parameter m for TQCPs. TQCPs (as red spheres) are between fully gapped topological superconductors (top) and trivial states (bottom) in a) and b). Majorana quantum fields form a representation of either π_{d+1}homotopy group of space-time Green's functions in a $(d+1)$ dimension surface when d is even (a)), or π_d homotopy group of zero-frequency Green's functions in a d-dimension surface embedded in $(d+1)$ dimension when d is odd (b)). c) TQCPs between gapless topological superconductors and gapped states. For time reversal symmetry breaking TQCPs, Quantum fields form a non-trivial class of π_{d-p}, the $(d-p)$th homotopy group of space-time Green's functions. $d-p(>0)$ is the dimension of a subspace where topological invariants are defined. $p = 0, 1$ are for a nodal point and nodal line phase respectively. In three dimension, only $p = 0$ nodal points are stable. d) is for time reversal invariant case. Quantum fields belong to non-trivial classes of π_{d-p-1}, the $(d-p-1)$th homotopy group of zero-frequency Green's function or invertible complex matrix $Q(\mathbf{k})$. Nodal points

What do we mean by saying “topological order” ?